Gravitational effects on and of vacuum decay

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(Received 4 March 1980)

It is possible for a classical field theory to have two stable homogeneous ground states, only one of which is an absolute energy minimum. In the quantum version of the theory, the ground state of higher energy is a false vacuum, rendered unstable by barrier penetration. There exists a well-established semiclassical theory of the decay of such false vacuums. In this paper, we extend this theory to include the effects of gravitation. Contrary to naive expectation, these are not always negligible, and may sometimes be of critical importance, especially in the late stages of the decay process.

I. INTRODUCTION

Consider the theory of a single scalar field defined by the action

$$S=\int d^4x[\tfrac{1}{2}(\partial_\mu\phi)^2-U(\phi)]\,, \tag{1.1}$$

where U is as shown in Fig. 1. That is to say, U has two local minima, $\phi_\pm$, only one of which, ϕ_-, is an absolute minimum. The classical field theory defined by Eq. (1.1) possesses two stable homogeneous equilibrium states, $\phi=\phi_+$ and $\phi=\phi_-$. In the quantum version of the theory, though, only the second of these corresponds to a truly stable state, a true vacuum. The first decays through barrier penetration; it is a false vacuum. This is a prototypical case; false vacuums occur in many field theories. In particular, they occur in some unified electroweak and grand unified theories, and it is this that gives the theory of vacuum decay possible physical importance. For simplicity, though, we will restrict ourselves here to the theory defined by Eq. (1.1); the extension of our methods to more elaborate field theories is straightforward.

The decay of the false vacuum is very much like the nucleation processes associated with first-order phase transitions in statistical mechanics.[1] The decay is initiated by the materialization of a bubble of true vacuum within the false vacuum. This is a quantum tunneling event, and has a certain probability of occurrence per unit time per unit volume, Γ/V. Once the bubble materializes, it expands with a speed asymptotically approaching that of light, converting false vacuum into true as it grows.

In the semiclassical (small $\hbar$) limit, Γ/V admits an expansion of the form

$$\Gamma/V=Ae^{-B/\hbar}[1+O(\hbar)]\,. \tag{1.2}$$

There exist algorithms for computing the coefficients A and B; indeed, in the limit of small energy-density difference between the two vacuums, it is possible to compute B in closed form. Also, in this same limit, it is possible to give a closed-form description of the growth of the bubble after its quantum formation. We will recapitulate this analysis later in this paper.

In this paper, we extend the theory of vacuum decay to include the effects of gravitation. At first glance, this seems a pointless exercise. In any conceivable application, vacuum decay takes place on scales at which gravitational effects are utterly negligible. This is a valid point if we are talking about the formation of the bubble, but not if we are talking about its subsequent growth. The energy released by the conversion of false vacuum to true is proportional to the volume of the bubble; thus, so is the Schwarzschild radius associated with this energy. Hence, as the bubble grows, the Schwarzschild radius eventually becomes comparable to the radius of the bubble.

This can easily be made quantitative. A sphere of radius Λ and energy density ϵ has Schwarzschild radius $2G\epsilon(4\pi\Lambda^3/3)$, where G is Newton's constant. This is equal to Λ when

$$\Lambda=(8\pi G\epsilon/3)^{-1/2}. \tag{1.3}$$

For an ϵ of $(1\text{ GeV})^4$, the associated Λ is 0.8 km. Of course, in a typical unified electroweak or grand unified field theory, the relevant energy densities are larger than this and the associated lengths correspondingly smaller. We are dealing here with phenomena which take place on scales neither subnuclear nor astronomical, but rather civic, or even domestic, scales far too small to neglect if we are interested in the cosmological consequences of vacuum decay. Contrary to naive expectation, the inclusion of gravitation is not point-

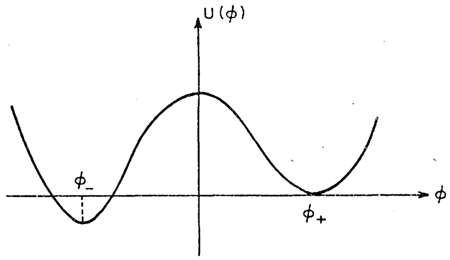

FIG. 1. The potential $U(\phi)$ for a theory with a false vacuum.

less; indeed, any description of vacuum decay that neglects gravitation is seriously incomplete.

Not only does gravitation affect vacuum decay, vacuum decay affects gravitation. In Eq. (1.1), there is no absolute zero of energy density; adding a constant to U has no effect on physics. This is not so when we include gravitation:

$$S=\int d^4x\sqrt{-g}\,[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-U(\phi)-(16\pi G)^{-1}R], \tag{1.4}$$

where R is the curvature scalar. Here, adding a constant to U is equivalent to adding a term proportional to $\sqrt{-g}$ to the gravitational Lagrangian, that is to say, to introducing a cosmological constant.[2] Thus, once the vacuum decays, gravitational theory changes; the cosmological constant inside the bubble is different from the one outside the bubble. Hence, in our computations, we need an initial condition not needed in the absence of gravitation. We must specify the initial value of the cosmological constant; equivalently, we must specify the absolute zero of energy density.

The experimental observation that the current value of the cosmological constant is zero gives two cases special interest. (1) $U(\phi_+)$ is zero. This would be the appropriate case to study if we were currently living in a false vacuum whose apocalyptic decay is yet to occur. (2) $U(\phi_-)$ is zero. This would be the appropriate case to study if we were living after the apocalypse, in the debris of a false vacuum which decayed at some early time in the history of the universe. Although our methods are applicable to arbitrary initial value of the cosmological constant, we pay special attention to these two cases.

The organization of the remainder of this paper is as follows: In Sec. II we summarize the theory of vacuum decay in the absence of gravitation. We emphasize the thin-wall approximation, the approximation that is valid in the limit of small energy-density difference between true and false vacuum. In Sec. III we begin the extension to gravitation, again emphasizing the thin-wall approximation. In the two special cases described above, we explicitly compute the effects of gravitation on the decay coefficient B of Eq. (1.2). As we have just argued, for any conceivable application, these effects are too small to worry about. Nevertheless, when we discovered it was within our power to compute them, we were unable to resist the temptation to do so. We have made no attempt to study the effects of gravitation on the coefficient A. This computation would involve the evaluation of a functional determinant; even if we had the courage to attempt such an evaluation, we would be frustrated by the nonrenormalizability of our theory. In Sec. IV we study the growth of the bubble. Section V states our conclusions. We have tried to write it in such a way that it will be intelligible to a reader who has skipped the intervening sections.

II. OLD RESULTS SUMMARIZED

In this section, we recapitulate the known results on vacuum decay in the theory defined by Eq. (1.1). The reader who wishes the arguments that lie behind our assertions is referred to the original literature.[1]

The Euclidean action is defined as minus the formal analytic continuation of Eq. (1.1) to imaginary time,

$$S_E=\int d^4x[\tfrac{1}{2}(\partial_\mu\phi)^2+U(\phi)], \tag{2.1}$$

where the metric is the usual positive-definite one of Euclidean four-space. The Euclidean equation of motion is the Euler-Lagrange equation associated with S_E. Let ϕ be a solution of this equation such that (1) ϕ approaches the false vacuum, ϕ_+, at Euclidean infinity, (2) ϕ is not a constant, and (3) ϕ has Euclidean action less than or equal to that of any other solution obeying (1) and (2). Then the coefficient B in the vacuum decay amplitude is given by

$$B=S_E(\phi)-S_E(\phi_+). \tag{2.2}$$

ϕ is called "the bounce." (The name has to do with the corresponding entity in particle mechanics.)

For the theories at hand, it can be shown[3] that the bounce is always O(4) symmetric, that is to say, ϕ is a function only of ρ, the Euclidean distance from an appropriately chosen center of coordinates. The Euclidean action then simplifies,

$$S_E=2\pi^2\int_0^\infty \rho^3 d\rho\,[\tfrac{1}{2}(\phi')^2+U], \tag{2.3}$$

as does the equation of motion

$$\phi''+\frac{3}{\rho}\phi'=\frac{dU}{d\phi}, \tag{2.4}$$

where the prime denotes $d/d\rho$.

It is possible to obtain an explicit approximation for ϕ in the limit of small energy-density difference between the two vacuums. Let us define ϵ by

$$\epsilon = U(\phi_+) - U(\phi_-) , \tag{2.5}$$

and let us write U as

$$U(\phi) = U_0(\phi) + O(\epsilon) , \tag{2.6}$$

where U_0 is a function chosen such that $U_0(\phi_-) = U_0(\phi_+)$, and such that $dU_0/d\phi$ vanishes at both ϕ_+ and ϕ_-.

The approximate ϕ obeys the equation

$$\phi'' = \frac{dU_0}{d\phi} . \tag{2.7}$$

Note that we have not only discarded the term in Eq. (2.4) proportional to ϵ, we have also discarded the term proportional to ϕ'. We will justify this shortly. Equation (2.7) admits a first integral,

$$[\tfrac{1}{2}(\phi')^2 - U_0]' = 0 . \tag{2.8}$$

Its value is determined by the condition that $\phi(\infty)$ is ϕ_+:

$$\tfrac{1}{2}(\phi')^2 - U_0 = -U_0(\phi_\pm) . \tag{2.9}$$

Thus, as ρ traverses the real line, ϕ goes monotonically from ϕ_- to ϕ_+. Equation (2.9) determines ϕ in terms of a single integration constant. We will choose this to be $\bar{\rho}$, the point at which ϕ is the average of its two extreme values:

$$\int_{(\phi_+ + \phi_-)/2}^{\phi} d\phi [2(U_0 - U_0(\phi_\pm)]^{-1/2} = \rho - \bar{\rho} . \tag{2.10}$$

Thus, for example, if

$$U_0 = \tfrac{1}{8}\lambda(\phi^2 - \mu^2/\lambda)^2 , \tag{2.11}$$

then

$$\phi = \frac{\mu}{\sqrt{\lambda}} \tanh[\tfrac{1}{2}\mu(\rho - \bar{\rho})] . \tag{2.12}$$

All that remains is to determine $\bar{\rho}$. We will do this on the assumption that $\bar{\rho}$ is large compared to the length scale on which ϕ varies significantly. For the example of Eq. (2.11), this means that $\bar{\rho}\mu$ is much greater than one. This assumption will be justified (for sufficiently small ϵ) at the end of the computation.

If $\bar{\rho}$ is large, the bounce looks like a ball of true vacuum, $\phi = \phi_-$, embedded in a sea of false vacuum, $\phi = \phi_+$, with a transition region ("the wall") separating the two. The wall is small in thickness compared to the radius of the ball; in our example, its thickness is $O(\mu^{-1})$. It is for this reason that the approximation we are describing is called "the thin-wall approximation." When we justify the thin-wall approximation, we will also justify our earlier neglect of the ϕ' term in Eq. (2.4). Away from the wall, this term is negligible because ϕ' is negligible; at the wall, it is negligible because $\bar{\rho}$ is large.

We will now determine $\bar{\rho}$ by computing B from Eq. (2.2) and demanding that it be stationary under variations of $\bar{\rho}$. The region of integration breaks naturally into three parts: outside the wall, inside the wall, and the wall itself. We divide B accordingly. Outside the wall, $\phi = \phi_+$. Hence,

$$B_{\text{outside}} = 0 . \tag{2.13}$$

Inside the wall, $\phi = \phi_-$. Hence,

$$B_{\text{inside}} = -\frac{\pi^2}{2}\bar{\rho}^4\epsilon . \tag{2.14}$$

Within the wall, in the thin-wall approximation,

$$\begin{aligned} B &= 2\pi^2\bar{\rho}^3 \int d\rho [\tfrac{1}{2}\phi'^2 + U_0(\phi) - U_0(\phi_+)] \\ &\equiv 2\pi^2\bar{\rho}^3 S_1 . \end{aligned} \tag{2.15}$$

Equation (2.9) gives us an integral expression for S_1,

$$S_1 = \int_{\phi_-}^{\phi_+} d\phi \{2[U_0(\phi) - U_0(\phi_+)]\}^{1/2} . \tag{2.16}$$

For future reference, we note that we can also write S_1 as

$$S_1 = 2 \int d\rho [U_0(\phi) - U_0(\phi_+)] . \tag{2.17}$$

We can now compute

$$B = -\tfrac{1}{2}\pi^2\bar{\rho}^4\epsilon + 2\pi^2\bar{\rho}^3 S_1 . \tag{2.18}$$

This is stationary at

$$\bar{\rho} = 3S_1/\epsilon . \tag{2.19}$$

We have justified our approximation: $\bar{\rho}$ indeed becomes large when ϵ becomes small. We now know

$$B = 27\pi^2 S_1^4/2\epsilon^3 . \tag{2.20}$$

We have used the bounce to compute a coefficient which enters into the probability for the quantum materialization of a bubble of true vacuum within the false vacuum. We can also use the bounce to describe the classical growth of the bubble after its materialization. The surface $t=0$ is the intersection of Euclidean space (imaginary time) and Minkowski space (real time). It can be shown that the value of ϕ on this surface can be thought of as the configuration of the field at the moment the bubble materializes. Also, at this moment, the time derivative of the field is zero. These initial-value data, together with the classical field equations in Minkowski space, suffice to determine the growth of the bubble.

Of course, there is no need to go to the bother of explicitly solving the classical field equations. All we need to do is analytically continue the Euclidean solution we already possess; that is to say, all we need to do is make the substitution

$$\rho \rightarrow (|\vec{x}|^2 - t^2)^{1/2}.$$

Thus, Euclidean O(4) invariance becomes Minkowskian O(3, 1) invariance. In the thin-wall approximation, the bubble materializes at rest with radius $\bar{\rho}$. As it grows, its surface traces out the hyperboloid $\rho = \bar{\rho}$. Since $\bar{\rho}$ is typically a quantity of subnuclear magnitude, this means that from the viewpoint of macrophysics, almost immediately upon its materialization the bubble accelerates to essentially the speed of light and continues to grow indefinitely at that speed.

III. INCLUSION OF GRAVITATION: MATERIALIZATION OF THE BUBBLE

In this section, we begin the extension of the analysis of Sec. II to the theory described by Eq. (1.4), the theory of a scalar field interacting with gravity.

As before, we begin by constructing a bounce, a solution of the Euclidean field equations obeying appropriate boundary conditions. This is apparently a formidable task; we now have to keep track of not just a single scalar field but also of the ten components of the metric tensor. However, things are not as bad as they seem, for there is no reason for gravitation to break the symmetries of the purely scalar problem. Thus it is reasonable to assume that, in the presence of gravity as in its absence, the bounce is invariant under four-dimensional rotations.

We emphasize that, in contrast to the case of a single scalar field, we have no theorem to back up this assumption. We will shortly construct, in the thin-wall approximation, an invariant bounce, but this will still leave open the possibility that there exist noninvariant bounces of lower Euclidean action. We do not think it likely that such objects exist, but we cannot prove they do not, and the reader should be warned that if they do exist, they dominate vacuum decay, and all our conclusions are wrong.

We begin by constructing the most general rotationally invariant Euclidean metric. The orbits of the rotation group are three-dimensional manifolds with the geometry of three-spheres. On each of these spheres, we introduce angular coordinates in the canonical way. We define a radial curve to be a curve of fixed angular coordinates. By rotational invariance, radial curves must be normal to the three-spheres through which they pass. We choose our radial coordinate ξ to measure distance along these radial curves. Thus, the element of length is of the form

$$(ds)^2 = (d\xi)^2 + \rho(\xi)^2 (d\Omega)^2 , \tag{3.1}$$

where $(d\Omega)^2$ is the element of distance on a unit three-sphere and ρ gives the radius of curvature of each three-sphere. Note that rotational invariance has made its usual enormous simplification; ten unknown functions of four variables have been reduced to one unknown function of one variable. Note also that we can redefine ξ by the addition of a constant without changing the form of the metric; equivalently, we can begin measuring ξ from wherever we choose.

Given Eq. (3.1), it is a straightforward exercise in the manipulation of Christoffel symbols to compute the Euclidean equations of motion. We will give only the results here. The scalar field equation is

$$\phi'' + \frac{3\rho'}{\rho}\phi' = \frac{dU}{d\phi}, \tag{3.2}$$

where the prime denotes $d/d\xi$. The Einstein equation

$$G_{\xi\xi} = -\kappa T_{\xi\xi}, \tag{3.3}$$

where $\kappa = 8\pi G$, becomes

$$\rho'^2 = 1 + \tfrac{1}{3}\kappa\rho^2(\tfrac{1}{2}\phi'^2 - U). \tag{3.4}$$

The other Einstein equations are either identities or trivial consequences of these equations. Finally,

$$S_E = 2\pi^2 \int d\xi \left(\rho^3(\tfrac{1}{2}\phi'^2 + U) + \frac{3}{\kappa}(\rho^2\rho'' + \rho\rho'^2 - \rho)\right). \tag{3.5}$$

In the thin-wall approximation, the construction of the bounce from these equations is astonishingly simple. Equation (3.2) differs from its counterpart in the pure scalar case, Eq. (2.4), in only two respects. Firstly, the independent variable is called ξ rather than ρ. This is a trivial change. Secondly, the coefficient of the ϕ' term involves a factor of ρ'/ρ rather than one of $1/\rho$. But this is also a trivial change, since in the thin-wall approximation we neglect this term anyway. (This is a bit facile. Of course, because the term is different in form, the eventual self-consistent justification of the approximation must also be different in form. We will deal with this problem when we come to it.) Thus, in the thin-wall approximation, we need only to copy Eq. (2.10),

$$\int_{(\phi_+ + \phi_-)/2}^{\phi} d\phi \{2[U_0 - U_0(\phi_\pm)]\}^{-1/2} = \xi - \bar{\xi}. \tag{3.6}$$

Here $\bar{\xi}$ is an integration constant, but, as we have explained, one with no convention-independent meaning.

Once we have ϕ, we can solve Eq. (3.4) to find ρ. This is a first-order differential equation; to specify its solution, we need one integration constant. We will choose this to be

$$\bar{\rho} \equiv \rho(\bar{\xi}). \tag{3.7}$$

This does have a convention-independent meaning; it is the radius of curvature of the wall separating false vacuum from true. We do not need the explicit expression for ρ for our immediate purposes, so we will not pause now to construct it.

Our next task is to find $\bar{\rho}$. The computation is patterned on that in Sec. II: First we compute B, the difference in action between the bounce and the false vacuum. Then we find $\bar{\rho}$ by demanding that B be stationary.

We first eliminate the second-derivative term from Eq. (3.5) by integration by parts; the surface term from the parts integration is harmless because we are only interested in the action difference between two solutions that agree at infinity. We thus obtain

$$S_E = 4\pi^2 \int d\xi \left(\rho^3(\tfrac{1}{2}\phi'^2 + U) - \frac{3}{\kappa}(\rho\rho'^2 + \rho)\right). \tag{3.8}$$

We now use Eq. (3.4) to eliminate ρ'. We find

$$S_E = 4\pi^2 \int d\xi \left(\rho^3 U - \frac{3\rho}{\kappa}\right). \tag{3.9}$$

So far, we have made no approximations. We now evaluate B, from Eq. (3.9), in the thin-wall approximation. As before, we divide the integration region into three parts. Outside the wall, bounce and false vacuum are identical; thus, as before,

$$B_{\text{outside}} = 0. \tag{3.10}$$

In the wall, we can replace ρ by $\bar{\rho}$, and U by U_0. Thus,

$$B_{\text{wall}} = 4\pi^2\bar{\rho}^3 \int d\xi\,[U_0(\phi) - U_0(\phi_+)] = 2\pi^2\bar{\rho}^3 S_1 \tag{3.11}$$

by Eq. (2.17). Inside the wall, ϕ is a constant. Hence,

$$d\xi = d\rho\,(1 - \tfrac{1}{3}\kappa\rho^2 U)^{-1/2} \tag{3.12}$$

and

$$B_{\text{inside}} = -\frac{12\pi^2}{\kappa}\int_0^{\bar{\rho}} \rho\, d\rho \{[1 - \tfrac{1}{3}\kappa\rho^2 U(\phi_-)]^{1/2} - (\phi_- \to \phi_+)\}$$
$$= \frac{12\pi^2}{\kappa^2}(U(\phi_-)^{-1}\{[1 - \tfrac{1}{3}\kappa\bar{\rho}^2 U(\phi_-)]^{3/2} - 1\} - (\phi_- \to \phi_+)). \tag{3.13}$$

[As a consistency check, it is easy to verify that this reduces to Eq. (2.14) when κ goes to zero.]

This is an ugly expression, and to continue our investigation in full generality would quickly involve us in a monstrous algebraic tangle. Thus we now restrict our attention to the two cases of special interest identified in Sec. I.

The first special case is decay from a space of positive energy density into a space of zero energy density. This is the case that is relevant if we are now in a postapocalyptic age. In this case

$$U(\phi_+) = \epsilon, \quad U(\phi_-) = 0. \tag{3.14}$$

It is then a trivial exercise to show that B is stationary at

$$\bar{\rho} = \frac{12 S_1}{4\epsilon + 3\kappa S_1^2} = \frac{\bar{\rho}_0}{1 + (\bar{\rho}_0/2\Lambda)^2}, \tag{3.15}$$

where $\bar{\rho}_0 = 3S_1/\epsilon$, the bubble radius in the absence of gravity, and $\Lambda = (\kappa\epsilon/3)^{-1/2}$, as in Eq. (1.3). At this point,

$$B = \frac{B_0}{[1 + (\bar{\rho}_0/2\Lambda)^2]^2}, \tag{3.16}$$

where $B_0 = 27\pi^2 S_1^4/2\epsilon^3$, the decay coefficient in the absence of gravity.

These equations have some interesting properties, but we will postpone discussing them until we write down the corresponding equations for the second special case. This is decay from a space of zero energy density into a space of negative energy density, the case that is relevant if we are now in a preapocalyptic age. In this case

$$U(\phi_+) = 0, \quad U(\phi_-) = -\epsilon. \tag{3.17}$$

As before, trivial algebra shows that

$$\bar{\rho} = \frac{\bar{\rho}_0}{1 - (\bar{\rho}_0/2\Lambda)^2} \tag{3.18}$$

and

$$B = \frac{B_0}{[1 - (\bar{\rho}_0/2\Lambda)^2]^2}. \tag{3.19}$$

These equations have been derived in the thin-wall approximation. Before we discuss their implications, we should discuss the reliability of the approximation. In the absence of gravitation, the thin-wall approximation was valid if $\bar{\rho}$ was large compared to the characteristic range of variation of ϕ; the significant quantity was $\bar{\rho}$ because $1/\rho$ multiplied the neglected ϕ' term in Eq. (2.4). In the presence of gravitation, $1/\rho$ is replaced by ρ'/ρ [see Eq. (3.2)]; thus it is this quantity that must be small at the wall.

By Eq. (3.4),

$$\frac{\rho'^2}{\rho^2}=\frac{1}{\rho^2}+\frac{\kappa}{3}\left(\tfrac{1}{2}\phi'^2-U\right). \tag{3.20}$$

The left-hand side of this equation is certainly small if both terms on the right are small. The first term is just $(1/\rho)^2$, as before. As for the second term, the quantity in parentheses is approximately constant over the wall, vanishes on one side of the wall, and has magnitude ϵ on the other. Thus, it is certainly an overestimation to replace it by ϵ everywhere; this turns the second term into $(1/\Lambda)^2$.

Thus, the thin-wall approximation is justified if both $\bar{\rho}$ and Λ are large compared to the characteristic range of variation of ϕ. This condition puts no restraint on $\bar{\rho}_0/\Lambda$, the ratio that measures the importance of gravitation; thus, it is not senseless to discuss our results for arbitrarily large values of this ratio. (Although it is not senseless, it is useless; as we said in Sec. I, in any conceivable application this ratio is negligible. We will proceed anyway.)

In the first special case, decay into the present condition, we see that gravitation makes the materialization of the bubble more likely (diminishes B), and makes the radius of the bubble at its moment of materialization smaller (diminishes $\bar{\rho}$). In the second special case, decay from the present condition, things are just the other way around; gravitation makes the materialization of the bubble less likely and its radius larger. Indeed, gravitation can totally quench vacuum decay; at $\rho_0=2\Lambda$ or, equivalently,

$$\epsilon=\tfrac{3}{4}\kappa S_1^{\,2}, \tag{3.21}$$

the bubble radius becomes infinite and the decay probability vanishes. For higher values of $\bar{\rho}_0$ or, equivalently, smaller values of ϵ, our equations admit of no sensible solution at all. Gravitation has stabilized the false vacuum.

We believe we understand this surprising phenomenon. Our explanation begins with a computation of the energy of a thin-walled bubble, in the absence of gravitation, at the time of its materialization. For the moment, we will give the bubble an arbitrary radius $\bar{\rho}$, postponing use of our knowledge that $\bar{\rho}$ is $\bar{\rho}_0$. The energy is the sum of a negative volume term and a positive surface term,

$$E=-\frac{4\pi}{3}\epsilon\bar{\rho}^3+4\pi S_1\bar{\rho}^2$$

$$=\frac{4\pi}{3}\epsilon\bar{\rho}^2(\bar{\rho}_0-\bar{\rho}). \tag{3.22}$$

We see that this vanishes for the actual bubble, $\bar{\rho}=\bar{\rho}_0$. This is as it should be. The energy of the world vanishes before the bubble materializes, and, whatever else barrier penetration may do, it does not violate the conservation of energy.

We now see how to compute the effects of gravitation on the bubble radius, in the limit that these effects are small. All we have to do is to compute the effects of gravitation on the total energy of the unperturbed bubble. If the gravitational contribution is positive, the bubble radius will have to grow, so Eq. (3.22) will develop a small negative contribution and total energy will remain zero. On the other hand, if the gravitational contribution is negative, the bubble will have to shrink. Of course, we already know that it is the former alternative that prevails, and not just in the limit of small gravitational effects. However, the point of the computation is to understand why it prevails.

There are two terms in the gravitational contribution to the energy of the unperturbed bubble. The first is the ordinary Newtonian potential energy of the bubble. This is easily computed by integrating over all space the square of the gravitational field, itself easily computed from Gauss's law. The answer is

$$E_{\text{Newton}}=-\epsilon\pi\bar{\rho}_0^{\,5}/15\Lambda^2. \tag{3.23}$$

Note that this is negative, as a gravitational potential energy should be. The second term comes from the fact that the nonzero energy density inside the bubble distorts its geometry. Thus, there is a correction to the volume of the bubble and thus to the volume term in the bubble energy. We can determine this correction to the volume from Eq. (3.12); the infinitesimal element of volume is

$$4\pi\rho^2 d\xi=4\pi\rho^2 d\rho\,(1-\tfrac{1}{2}\rho^2/\Lambda^2)+O(G^2). \tag{3.24}$$

Note that this is smaller than the Euclidean formula; thus this geometrical correction reduces the magnitude of the negative volume energy and hence makes a positive contribution to the total energy. Integration yields

$$E_{\text{geom}}=2\pi\epsilon\bar{\rho}_0^{\,5}/5\Lambda^2 \tag{3.25}$$

in the small-G limit. The total gravitational correction is the sum of these two terms,

$$E_{\text{grav}}=\pi\epsilon\bar{\rho}_0^{\,5}/3\Lambda^2. \tag{3.26}$$

This is positive; hence, the bubble is larger in the presence of gravitation than in its absence.

We now understand what is happening. Vacuum decay proceeds by the materialization of a bubble. By energy conservation, this bubble always has energy zero, the sum of a negative volume term and a positive surface term. In the absence of gravity, we can always make a zero-energy bubble no matter how small ϵ is; we just have to make the bubble large enough, and the volume/surface ration will do the job. However, in the presence of gravity, the negative energy density inside the bubble distorts the geometry of space in such a way as to diminish the volume/surface ratio. Thus it is possible that, for sufficiently small ϵ, no bubble, no matter how big, will have energy zero. What Eq. (3.18) is telling us is that this is indeed the case.

We cannot exclude the possibility that decay could proceed through nonspherical bubbles, although we think this is unlikely; we guess that such configurations would only worsen the volume/surface ratio. Nor can we exclude the possibility that decay proceeds through some nonsemiclassical process, one that does not involve bubble formation at all, and whose probability vanishes more rapidly than exponentially in the small-$\hbar$ limit. About this possibility we cannot even make guesses.

IV. INCLUSION OF GRAVITATION: GROWTH OF THE BUBBLE

In Sec. II, we explained how to obtain a description of the classical growth of the bubble after its quantum materialization, in the absence of gravitation. All we had to do was analytically continue the scalar field ϕ from Euclidean space to Minkowskian space. Because of the enormous symmetry of the bounce—O(4)-invariant in Euclidean space, O(3, 1)-invariant in Minkowski space—for much of Minkowski space this continuation was trivial. To be more precise, if we choose the center of the bubble at its moment of materialization to be the center of coordinates, for all spacelike points the contribution was a mere reinterpretation of ρ as Minkowskian spacelike separation rather than Euclidean radial distance. It was only for timelike points that nontrivial continuation (to imaginary ρ) was needed.

All of this carries over to the case in which gravitation is present. The only difference is that we have to continue the metric as well as the scalar field, turning an O(4)-invariant Euclidean manifold into an O(3, 1)-invariant Minkowskian manifold.

Thus, a large part of the manifold is analogous to the spacelike region described above. Here,

$$ds^2 = -d\xi^2 - \rho(\xi)^2 (d\Omega_S)^2 , \tag{4.1}$$

where $d\Omega_S$ is the element of length on a unit hyperboloid with spacelike normal vector in Minkowski space. [The overall minus sign has appeared because we adhere to the convention that a Minkowski metric has signature $(+---)$.] In this region, ϕ is $\phi(\rho)$. In the thin-wall approximation, the bubble wall always lies within this region at $\rho=\bar{\rho}$. Thus, this is all the manifold we need if we are only interested in studying the expansion of the bubble as it appears from the outside.

However, if we wish to go inside the bubble, we may encounter vanishing ρ. This is a pure coordinate singularity analogous to reaching the light cone, the boundary of the spacelike region, in Minkowski space. We get beyond the singularity by continuing to the timelike region. We choose ξ to be zero when ρ is zero, and continue to $\xi = i\tau$, with τ real. We thus obtain

$$ds^2 = d\tau^2 - \rho(i\tau)^2 (d\Omega_T)^2 , \tag{4.2}$$

where $d\Omega_T$ is the element of length for a unit hyperboloid with timelike normal in Minkowski space. One way of describing this equation is to say that the interior of the bubble always contains a Robertson-Walker universe of open type.

Let us now apply this general prescription to the special cases we have analyzed in Sec. III. In the thin-wall approximation, no analytic continuation is needed for ϕ; it is equal to ϕ_+ outside the bubble and ϕ_- inside the bubble. The metric outside the bubble is obtained by solving Eq. (3.4):

$$\rho'^2 = 1 - \frac{\kappa\rho^2}{3} U(\phi_+) . \tag{4.3}$$

Inside the bubble, it is obtained by solving the same equation with ϕ_+ replaced by ϕ_-. The two metrics are joined at the bubble wall, not at the same ξ, but at the same ρ, $\rho=\bar{\rho}$.

In our first special case, decay into the present condition, U vanishes inside the bubble. Thus the interior metric is $\rho=\xi$, ordinary Minkowski space. Outside the bubble, Eq. (4.3) becomes

$$\rho'^2 = 1 - \rho^2/\Lambda^2 . \tag{4.4}$$

The solution is

$$\rho = \Lambda \sin(\xi/\Lambda) . \tag{4.5}$$

We shall now show that this is ordinary de Sitter space, written in slightly unconventional coordinates.[4]

We begin by recapitulating the definition of de Sitter space. Consider a five-dimensional Minkowski space with O(4, 1)-invariant metric

$$ds^2 = -(dw)^2 + (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 . \tag{4.6}$$

In this space, consider the hyperboloid defined by

$$\Lambda^2 = w^2 - t^2 + x^2 + y^2 + z^2 , \tag{4.7}$$

with Λ some positive number. This is a four-dimentional manifold with a Minkowskian metric; it is de Sitter space. Note that de Sitter space is as homogeneous as Minkowski space; any point in the space can be transformed into any other by an O(4, 1) transformation.

To put the metric of de Sitter space into our standard form, we must choose the location of the center of the bubble at its moment of materialization. Since de Sitter space is homogeneous, we can without loss of generality choose this point to be $(\Lambda, 0, 0, 0, 0)$. The O(3, 1) group of the vacuum decay problem is then the Lorentz group acting on the last four coordinates. Thus we replace these by "angular" coordinates, as in Eq. (4.1):

$$ds^2 = -(dw)^2 - (d\rho)^2 - \rho^2 (d\Omega_S)^2 . \tag{4.8}$$

Equation (4.7) then becomes

$$\Lambda^2 = w^2 + \rho^2 .$$

If we now define ξ by

$$w = \Lambda \cos(\xi/\Lambda), \quad \rho = \Lambda \sin(\xi/\Lambda) , \tag{4.9}$$

the metric falls into the desired form.

In this metric, ρ is bounded above by Λ. The geometrical reason for this is clear from Eq. (4.7). A spacelike slice of de Sitter space (say the hypersurface $t=0$) is a hypersphere of radius Λ; on a hypersphere, no circle has greater circumference than a great circle. This also explains a curious feature of Eq. (3.15),

$$\bar{\rho} = \frac{\bar{\rho}_0}{1 + (\bar{\rho}_0/2\Lambda)^2} . \tag{3.15}$$

No matter how we choose $\bar{\rho}_0$, $\bar{\rho}$ is always less than or equal to Λ. The reason for this is now obvious; the bubble cannot be bigger than this because a bigger bubble could not fit into the false vacuum.

We now go to our second special case, the decay of the present vacuum. Here U vanishes outside the bubble, so it is the exterior metric that is ordinary Minkowski space. Inside the bubble, Eq. (4.3) becomes

$$\rho'^2 = 1 + \rho^2/\Lambda^2 . \tag{4.10}$$

The solution is

$$\rho = \Lambda \sinh(\xi/\Lambda) . \tag{4.11}$$

Since we are now inside the bubble, we will also need the continuation of this to the timelike region, the Robertson-Walker universe inside the bubble. By Eq. (4.2) this is

$$ds^2 = d\tau^2 - \Lambda^2 \sin^2(\tau/\Lambda)[d\Omega_T]^2 . \tag{4.12}$$

This is an open expanding-and-contracting universe. We normally think of oscillating universes as necessarily closed, but this rule depends on the positivity of energy, very much violated here.

The metric defined by Eq. (4.12) has singularities when τ is an integral multiple of $\pi\Lambda$. We shall now show that these singularities are spurious, mere coordinate artifacts.

Consider a five-dimensional Minkowski space with O(3, 2)-invariant metric

$$ds^2 = (dw)^2 + (dt)^2 - (dx)^2 - (dy)^2 - (dz)^2 . \tag{4.13}$$

In this space, consider the hyperboloid defined by

$$\Lambda^2 = w^2 + t^2 - x^2 - y^2 - z^2 , \tag{4.14}$$

with Λ some positive number. This is a four-dimensional manifold with a Minkowski metric; indeed, by applying the same coordinate transformations we used in our analysis of de Sitter space, one can easily show that the metric is that defined by Eqs. (4.11) and (4.12).

Although the hyperboloid is free of singularities, it is not totally without pathologies; it contains closed timelike curves, for example, circles in the w-t plane. These can easily be eliminated. The hyperboloid is homeomorphic by $R_3 \times S^1$. (That is to say, once we have freely given x, y, and z, w and t must lie on a circle.) Thus, it is not simply connected, and we may replace it by its simply connected covering space. [In Eq. (4.12), this corresponds to interpreting τ as an ordinary real variable rather than an angular variable.] This covering has neither singularities nor closed timelike curves. It is much like de Sitter space, except that its symmetry group is O(3, 2) rather than O(4, 1); it is called anti–de Sitter space.[5]

Anti–de Sitter space is the universe inside the bubble. We shall now show that this universe is dynamically unstable; even the tiny corrections to the thin-wall approximation are sufficient to convert the coordinate singularity in Eq. (4.12) into a genuine singularity, to cause gravitational collapse.

For our discussion, we need the exact field equations in the timelike region. These are

$$\ddot{\phi} + \frac{3\dot{\rho}}{\rho}\dot{\phi} + \frac{dU}{d\phi} = 0 \tag{4.15}$$

and

$$\dot{\rho}^2 = 1 + \frac{\kappa\rho^2}{3}(\tfrac{1}{2}\dot{\phi}^2 + U) , \tag{4.16}$$

where the dot indicates differentiation with respect to τ. In general, initial-value data for this system consists of a point in the ϕ-$\dot{\phi}$ plane at some value of ρ. (The associated value of τ has no convention-independent meaning.) In the special case of vanishing ρ, a nonsingular solution must have vanishing $\dot{\phi}$, just as, in Euclidean space, a rotationally invariant function must have vanishing gradient at the origin of coordinates.

For notational simplicity in what follows, we choose the zero of ϕ to be the true vacuum $\phi_- = 0$. Also, we define

$$\mu^2 \equiv \left.\frac{d^2U}{d\phi^2}\right|_0 . \tag{4.17}$$

Until now, we took the initial-value data at $\rho = \tau = 0$ to be $\phi = \dot{\phi} = 0$. This led to the solution

$$\phi = 0 , \tag{4.18}$$

and

$$\rho = \Lambda \sin\tau/\Lambda . \tag{4.19}$$

Now, vanishing $\dot{\phi}$ at $\rho = 0$ is an exact result; the bounce is rotationally invariant. But vanishing ϕ is just an approximation; as can be seen from the explicit formula for the bounce, at $\rho = 0$, ϕ is $O(\exp[-\mu\rho])$, exponentially small but not zero.

As long as ϕ remains exponentially small, we can neglect its effects on ρ and continue to use Eq. (4.19). Also, we can replace Eq. (4.15) by its linear approximation

$$\ddot{\phi} + \frac{3\dot{\rho}}{\rho}\dot{\phi} + \mu^2\phi = 0 . \tag{4.20}$$

Of course, if in the course of time ϕ grows large, we can no longer make these approximations and must return to the exact equations.

We begin by solving Eq. (4.20) for $\rho/\Lambda \ll 1$. In this region, ρ is approximately τ, anf Eq. (4.20) is a Bessel equation. Thus, ϕ is simply an exponentially small multiple of the nonsingular oscillatory solution of this equation and remains exponentially small throughout this region.

We next turn to the region $\rho\mu \gg 1$. Note that under the conditions of the thin-wall approximation this overlaps the preceding region. In this region, Eq. (4.20) is the Newtonian equation of motion for a weakly damped harmonic oscillator, with a slowly varying damping coefficient $3\dot{\rho}/\rho$. Thus,

$$\phi \propto \cos(\mu\tau + \theta) \exp\left(-\int d\tau\, 3\dot{\rho}/2\rho\right) \tag{4.21}$$

or

$$\phi = a\rho^{-3/2} \cos(\mu\tau + \theta) . \tag{4.22}$$

Here a is an exponentially small coefficient and θ is an angle independent of any of the parameters of the theory, whose explicit value is of no interest to us. We see that ϕ remains exponentially small throughout the expansion and subsequent contraction of the universe, all the way down to the region where ρ/Λ is once again much less than one. In this region, we may rewrite Eq. (4.22) as

$$\phi = a\rho^{-3/2} \cos(\pi\Lambda\mu - \rho\mu + \theta) . \tag{4.23}$$

If we attempt to continue all the way to the second zero of ρ predicted by Eq. (4.19), ϕ becomes large and our approximations break down. However, whatever happens to ϕ, a second zero is inevitable. From Eq. (4.16) and the assumed properties of U,

$$\dot{\rho}^2 \geq 1 - \rho^2/\Lambda^2 . \tag{4.24}$$

Thus, once ρ is much less than Λ and diminishing, it must continue to diminish all the way to zero, and it must do so quickly, in a time of order ρ.

Of course, vanishing ρ does not imply singular behavior. On the contrary, we can always obtain a one-parameter family of nonsingular solutions by integrating the equations of motion backward from the second zero of ρ, using as final-value data any point on the line $\dot{\phi} = 0$. If we continue this family of solutions into the region of validity of Eq. (4.23), then, for any fixed ρ, they will define a curve in the ϕ-$\dot{\phi}$ plane. Because we are integrating the equations of motion over a very short time interval, this curve can have only a negligible dependence on the small parameter ϵ, the energy-density difference between the two vacuums. On the other hand, the angle in the ϕ-$\dot{\phi}$ plane obtained from Eq. (4.23) at fixed ρ is a very rapidly varying function of ϵ because Λ is proportional to $\epsilon^{-1/2}$.

Thus for general ϵ, Eq. (4.23) does not continue to a nonsingular solution. This argument does not eliminate the possibility that there might be special values of ϵ for which the singularity may be avoided. However, we believe this possibility is of little interest; an instability that can be removed only by fine tuning the parameters of the theory is going to return once we consider other corrections to our approximation (e.g., the effects of a small initial matter density). This completes the argument for gravitational collapse of the bubble interior.

V. CONCLUSIONS

Although some of the results in the body of this paper hold in more general circumstances, we will restrict ourselves here to the thin-wall approximation, the approximation that is valid in the limit of small energy-density difference between true and false vacuum, and to the two cases of special interest identified in Sec. I.

The first special case is decay from a space of vanishing cosmological constant, the case that applies if we are currently living in a false vacuum. In the absence of gravitation, vacuum decay proceeds through the quantum materialization of a bubble of true vacuum, separated by a thin wall from the surrounding false vacuum. The bubble is at rest at the moment of materialization, but it rapidly grows; its wall traces out a hyperboloid in Minkowski space, asymptotic to the light cone.

If all we are interested in is vacuum decay as seen from the outside, not a word of the preceding description needs to be changed in the presence of gravitation. At least at first glance, this is surprising because one would imagine that gravitation would have some effect on the growth of the bubble. There are two ways of understanding why this does not happen. The first is a mathematical way: The growth of the bubble in the absence of gravitation is O(3, 1) invariant; the inclusion of gravitation does not spoil this invariance; the only O(3, 1)-invariant hypersurfaces are hyperboloids with light cones as their asymptotes. The second is a physical way: Quantum tunneling does not violate the law of conservation of energy; thus the total energy of the expanding bubble is always identically zero, the negative energy of the interior being canceled by the positive energy of the wall. Because the bubble is spherically symmetric, the gravitational field at the outer edge of the wall is determined exclusively by the total energy within. That is to say, it is zero, and neither accelerates nor retards the growth of the edge.

Of course, gravitation affects the quantitative features of vacuum decay. In any conceivable application, these effects are totally negligible, but we have computed them anyway. In general, gravitation makes the probability of vacuum decay smaller; in the extreme case of very small energy-density difference, it can even stabilize the false vacuum, preventing vacuum decay altogether. We believe we understand this. For the vacuum to decay, it must be possible to build a bubble of total energy zero. In the absence of gravitation, this is no problem, no matter how small the energy-density difference; all one has to do is make the bubble big enough, and the volume/surface ratio will do the job. In the presence of gravitation, though, the negative energy density of the true vacuum distorts geometry within the bubble with the result that, for a small enough energy density, there is no bubble with a big enough volume/surface ratio.

Within the bubble, the effects of gravitation are more dramatic. The geometry of space-time within the bubble is that of anti–de Sitter space, a space much like conventional de Sitter space except that its group of symmetries is O(3, 2) rather than O(4, 1). Although this space-time is free of singularities, it is unstable under small perturbations, and inevitably suffers gravitational collapse of the same sort as the end state of a contracting Friedmann universe. The time required for the collapse of the interior universe is on the order of the time Λ discussed in Sec. I, microseconds or less.

This is disheartening. The possibility that we are living in a false vacuum has never been a cheering one to contemplate. Vacuum decay is the ultimate ecological catastrophe; in a new vacuum there are new constants of nature; after vacuum decay, not only is life as we know it impossible, so is chemistry as we know it. However, one could always draw stoic comfort from the possibility that perhaps in the course of time the new vacuum would sustain, if not life as we know it, at least some structures capable of knowing joy. This possibility has now been eliminated.

The second special case is decay into a space of vanishing cosmological constant, the case that applies if we are now living in the debris of a false vacuum which decayed at some early cosmic epoch. This case presents us with less interesting physics and with fewer occasions for rhetorical excess than the preceding one. It is now the interior of the bubble that is ordinary Minkowski space, and the inner edge of the wall that continues to trace out a hyperboloid. The mathematical reason for this is the same as before. The physical reason is even simpler than before: Within a spherically symmetric shell of energy there is no gravitational field.

As before, the effects of gravitation are negligible in any conceivable application, but we have computed them anyway. The sign is opposite to that in the previous case; gravitation makes vacuum decay more likely.

The space-time outside the bubble is now conventional de Sitter space. Of course, neither this space nor the Minkowski space inside is subject to catastrophic gravitational collapse initiated by small perturbations.

Finally, we must comment on the problem of the cosmological constant in the context of spontaneous symmetry breakdown. This problem was raised some years ago[2] and we have little new to say about it, but our work here has brought it home to us with new force.

Normally, when something is strictly zero, there is a reason for it. Vector-meson squared masses vanish because of gauge invariance, those of Dirac fields because of chiral symmetry. There is nothing to keep the squared masses of scalar mesons zero, but it is no disaster if they go negative, merely a sign that we are expanding about the wrong ground state.

But there is no reason for the cosmological constant (equivalently, the absolute energy density of the vacuum) to vanish. Indeed, if it were not for the irrefutable empirical evidence, one would expect it to be a typical microphysical number, the radius of the universe to be less than a kilometer. Even worse, zero energy density is the edge of disaster; even the slightest negativity would be enough to initiate catastrophic gravitational collapse.

There is something we do not understand about gravitation, and this something has nothing to do with loops of virtual gravitons. There has to be change, and change at a length scale much larger than the Planck length.

ACKNOWLEDGMENTS

This work was supported in part by the Department of Energy under Contract No. DE-AC03-76SF00515 and Contract No. DE-AS02-76ER02220 and by the National Science Foundation under Grant No. PHY77-22864.

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[1]We follow here the treatment given in S. Coleman, Phys. Rev. D 15, 2929 (1977); 16, 1248(E) (1977); and C. G. Callan and S. Coleman, *ibid.* 16, 1762 (1977). These contain references to the earlier literature.

[2]A. Linde, Pis'ma Zh. Eksp. Teor. Fiz. 19, 320 (1974) [JETP Lett. 19, 183 (1974)]; M. Veltman, Phys. Rev. Lett. 34, 777 (1975).

[3]S. Coleman, V. Glaser, and A. Martin, Commun. Math. Phys. 58, 211 (1978).

[4]Our treatment of de Sitter space closely follows that of S. Hawking and G. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, New York, 1973).

[5]For more on anti–de Sitter space, see Hawking and Ellis (Ref. 4).