1 General comments

1.1 Applications

- Deposition of thin films in semiconductor manufacture.
- Formation of snowflakes.

1.2 Models and modeling

A model is a **representation** of a physical system, invariably **idealized** to a greater or lesser extent.

Examples:

- Newton's second law
$$F = ma$$
It assumes the existence of *inertial reference frames*. It also assumes that the *mass* of a body, m, is invariant under coordinate transformations.
- A wind tunnel.
- Old Japanese Godzilla movies and *Finding Nemo*. Filming Godzilla walking tall beside a miniature skyscraper looks fake because the scaled down representation does not adequately *model* the 'reality' of a giant monster wreaking havoc among tall buildings.

Are there phenomena that cannot be modeled effectively? The weather? Climate change? What does 'effectively' mean?

1.3 Discrete versus continuous models

It is generally harder to deduce the properties of *discrete mathematical systems* than it is for systems in which continuity and/or differentiability may be assumed. *Continuity* and *differentiability* are powerful constraints which sometimes permit strong, global conclusions. Recall, for example

- *Intermediate value theorem*: Let f be a function which is continuous on the closed interval $[a, b]$. Suppose that d is a real number between $f(a)$ and $f(b)$; then there exists c in $[a, b]$ such that $f(c) = d$.

- *Rolle's Theorem*: Let f be a function which is differentiable on the closed interval $[a, b]$. If $f(a) = f(b)$ then there exists a point c in (a, b) such that $f'(c) = 0$.

1.4 Examples of discrete and continuous models in population dynamics

- *Continuous time and continous space.* Human population dynamics. Because the human population is large, birth and death occur nearly continuously in time.

- *Discrete time, continuous space.* Some species of trees. A maple tree, for example, releases all its seeds over a relatively short period once per year.

- *Discrete space, continuous time.* Patch models. Populations of animals isolated into subpopulations by water, mountains or man-made structures such as roads or agricultural fields.

2 Random deposition model

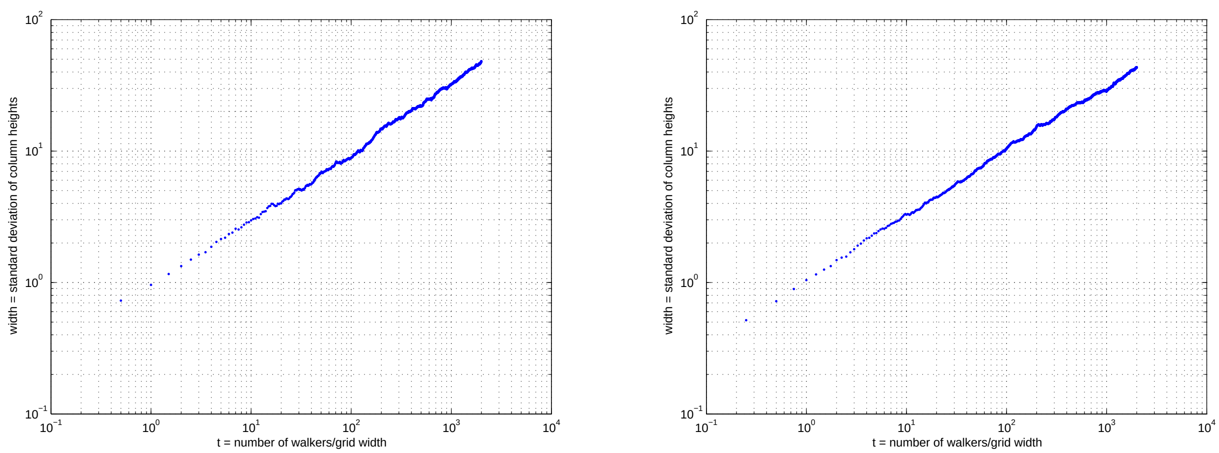
At a position chosen at random over a growing one-dimensional surface, introduce a *walker* (or *particle*) and allow it to fall vertically. When it reaches the top of the column below it, capture it.

Under the random deposition model, the growth of the interface is **uncorrelated**, that is, the columns of captured walkers grow independently. This is *not true* for an interface that grows by *ballistic deposition* to be discussed later.

How does the "roughness" of the surface built by a process modeled by random deposition vary as more and more walkers are captured?

2.1 Results of numerical experiments

Figure 1 shows the increase of the interface width as a function of time under random deposition for two different system sizes. The time t is given in units of walkers per system width.



(a) L = 100, n = 200,000

(b) L = 200, n = 400,000

Figure 1: Interface width as a function of time for random deposition

2.2 The binomial distribution

Phenomena modeled by a binomial distribution include

- A fair or biased coin tossed n times. The probability of observing 1 head, 2 heads, m heads ($m \leq n$) is given by the binomial distribution.
- A particle that moves *one step to the right* with probability p or *one step to the left* with probability q ($p + q = 1$) n times. The probability that the particle moves a distance d from the origin after n steps is governed by the binomial distribution.
- The column heights under random deposition are binomially distributed.

2.2.1 Derivation of the binomial distribution

Consider a walker starting at the origin that moves one step to the left or right with fixed probabilities

$$P(+1) = p,$$
$$P(-1) = q = 1 - p.$$

at every time step.

Question: What is the probability that the walker will have moved r steps to the right of the origin after a fixed number of steps, n?

- **Formulation**: Assume the walker moves r steps to the right and l steps to the left in a total of $n = r + l$ steps. Note: such a walker will have moved a distance $d = r - l$ from the origin in n steps.
- **Count permutations**: [Draw $n = r + l$ 'containers' into which to place r's and l's] There are n ways to 'sequence' the first 'r' move, $n - 1$ ways to sequence the second, etc. Once all the 'r' moves are placed, the sequence of 'l' moves is determined.

$$n(n-1)...(n-r+1) = \frac{n!}{(n-r)!}.$$

- **Count combinations**: To find the number of *combinations*, we reduce the count of states with probability $p^r q^l$ by the number of such **repeated** or **indistinguishable** states. Each configuration of 'r' moves includes $r!$ repeats.

- **Binomial distribution**: The probability of moving 'r' steps to the right in n steps is therefore

$$P_n(r) = \frac{n!}{r!(n-r)!} p^r q^l,$$
$$= \frac{n!}{r!(n-r)!} p^r q^{n-r}.$$

  This is the **binomial distribution**.

2.2.2 Verify that this is a probability distribution

For the preceding to be a **probability distribution**, $P_n(r)$ must satisfy

$$\sum_{r=0}^{n} P_n(r) = 1.$$

But this is just the expansion of the binomial $(a+b)^n$ with $a = p$ and $b = q$ and $p + q = 1$.

2.2.3 The expected value

The **expected value** for the quantity $f(x)$ where x has the (discrete) probability distribution $P(x)$ is given by

$$E[f(x)] = \sum_x f(x)P(x).$$

In particular,

$$E(x) = \sum_x xP(x).$$

For the binomial distribution, we have for the expected value of r (the number of steps taken to the right)

$$E(r) = \sum_{r=0}^{n} r \frac{n!}{r!(n-r)!} p^r q^{n-r}.$$

But at $r = 0$, the summand is zero so we may as well rewrite the summation as

$$\begin{aligned}
E(r) &= \sum_{r=1}^{n} r \frac{n!}{r!(n-r)!} p^r q^{n-r}, \\
&= \sum_{r=1}^{n} \frac{n!}{(r-1)!(n-r)!} p^r q^{n-r}.
\end{aligned}$$

Let $\overline{r} = r - 1$. Then

$$\begin{aligned}
E(r) &= \sum_{\overline{r}=0}^{n-1} \frac{n!}{\overline{r}!(n-\overline{r}-1)!} p^{\overline{r}+1} q^{n-\overline{r}-1}, \\
&= pn \sum_{\overline{r}=0}^{n-1} \frac{(n-1)!}{\overline{r}![(n-1)-\overline{r}]!} p^{\overline{r}} q^{[(n-1)-\overline{r}]}, \\
&= pn \sum_{\overline{r}=0}^{n-1} P_{n-1}(\overline{r}), \\
&= pn.
\end{aligned}$$

2.2.4 The variance

The **variance** measures the squared "width" of a distribution, i.e., how closely a distribution is clustered about its expected value.

$$V(r) = E([r - E(r)]^2).$$

The **standard deviation** measures the "width".

$$s(r) = \sqrt{V(r)}.$$

Note the following important result.

$$\begin{aligned}
E[(r - E(r))^2] &= \sum (r - E(r))^2 P(r), \\
&= \sum [r^2 - 2rE(r) + (E(r))^2] P(r), \\
&= \sum r^2 P(r) - 2E(r) \sum rP(r) + (E(r))^2 \sum P(r), \\
&= E(r^2) - 2(E(r))^2 + (E(r))^2, \\
&= E(r^2) - (E(r))^2.
\end{aligned}$$

To compute the variance of the binomial distribution, we need $E(r^2)$.

$$\begin{aligned} E(r^2) &= \sum_{r=0}^{n} r^2 \frac{n!}{r!(n-r)!} p^r q^{n-r}, \\ &= \sum_{r=1}^{n} r^2 \frac{n!}{r!(n-r)!} p^r q^{n-r}, \\ &= \sum_{r=1}^{n} r \frac{n!}{(r-1)!(n-r)!} p^r q^{n-r}. \end{aligned}$$

As before, let $\overline{r} = r - 1$. Find that

$$\begin{aligned} E(r^2) &= \sum_{\overline{r}=0}^{n-1} (\overline{r}+1) \frac{n!}{\overline{r}!(n-\overline{r}-1)!} p^{\overline{r}+1} q^{n-\overline{r}-1}, \\ &= np \sum_{\overline{r}=0}^{n-1} (\overline{r}+1) \frac{(n-1)!}{\overline{r}!((n-1)-\overline{r})!} p^{\overline{r}} q^{(n-1)-\overline{r}}, \\ &= np[E(P_{n-1}(r)) + 1] \\ &= np[(n-1)p + 1]. \end{aligned}$$

Therefore,

$$\begin{aligned} E[(r - E(r))^2] &= E(r^2) - (E(r))^2, \\ &= np[(n-1)p + 1] - (np)^2, \\ &= np[1 - p]. \end{aligned}$$

The variance is

$$V(r) = npq$$

and the standard deviation is

$$s(r) = \sqrt{npq}.$$

2.3 Results

2.3.1 The mean height of the growing interface under random deposition

Let L be the width of the grid in *number of columns.*

Under the *random deposition* capture model, with walkers introduced uniformly across the breadth of the grid and with walkers permitted only to fall, the height of the columns is a binomially distributed random variable

$$P_n(h) = \frac{n!}{h!(n-h)!} p^h q^{n-h}.$$

Here n is the number of 'trials' (walkers introduced to the grid) and $p = 1/L$ is the probability of a walker being added to a particular column.

The *mean column height* under random deposition is the expected value of h under the binomial distribution.

$$E(h) = np = n/L$$

In your computational model, compute the mean height of the columns (the mean surface height), $\overline{h}$, as

$$\overline{h}(t) = \frac{1}{L} \sum_{i=1}^{L} h(i,t).$$

where $h(i,t)$ is the height of column i at time t. Use $t = n$, the number of walkers simulated, for the time.

2.3.2 The width of the growing interface under random deposition

Under random deposition, the height of each column evolves independently of the others. The standard deviation of the column heights is $\sqrt{npq}$ with $p = 1/L$ and $q = 1 - p$. Consequently, as the number of walkers n increases, the **width of the interface also increases as** $\sqrt{npq} \sim \sqrt{n}$. However, the *relative width* of the interface decreases as

$$\begin{aligned} \frac{\sqrt{npq}}{np} &\sim \frac{\sqrt{n}}{n}, \\ &\sim \frac{1}{\sqrt{n}}. \end{aligned}$$

In your computational model, compute the interface width as

$$w(L,t) = \sqrt{\frac{1}{L} \sum_{i=1}^{L} [h(i,t) - \overline{h}(t)]^2}$$

where $\overline{h}(t)$ is the mean height of the interface at time t.

2.4 The expected value and variance of the left/right distance travelled under a symmetric random walk

If you run your simulation using the models *no interaction*, *symmetric random walk*, *center walker generator*, *absorbing boundary conditions* and a sufficiently small grid aspect ratio (height/width), you will find that the captured walkers form a bell-shaped distribution centered at the horizontal point of introduction of the walkers.

A particle undergoing a left/right symmetric random walk travels a net horizontal distance $d = r - l$, where r is the number or right steps taken and l is the number of left steps.

The **expected value** of d is given by

$$\begin{aligned} E(d) &= E(r - l), \\ &= np - nq, \\ &= n(p - q). \end{aligned}$$

If $p = q$, $E(d) = 0$.

The **variance** of the distance travelled by a walker undergoing a left/right symmetric random walk is given by

$$\begin{aligned} V(d) &= E[(d - E(d))^2], \\ &= E(d^2) - [E(d)]^2. \end{aligned}$$

$d = r - l$ implies that $d = r - (n - r)$ or $d = 2r - n$. Therefore,

$$\begin{aligned} E(d^2) &= E[4r^2 - 4nr + n^2], \\ &= 4E(r^2) - 4nE(r) + n^2, \\ &= 4[np((n-1)p + 1)] - 4n(np) + n^2, \\ &= n^2 - 4n(n-1)p(1-p), \end{aligned}$$

and

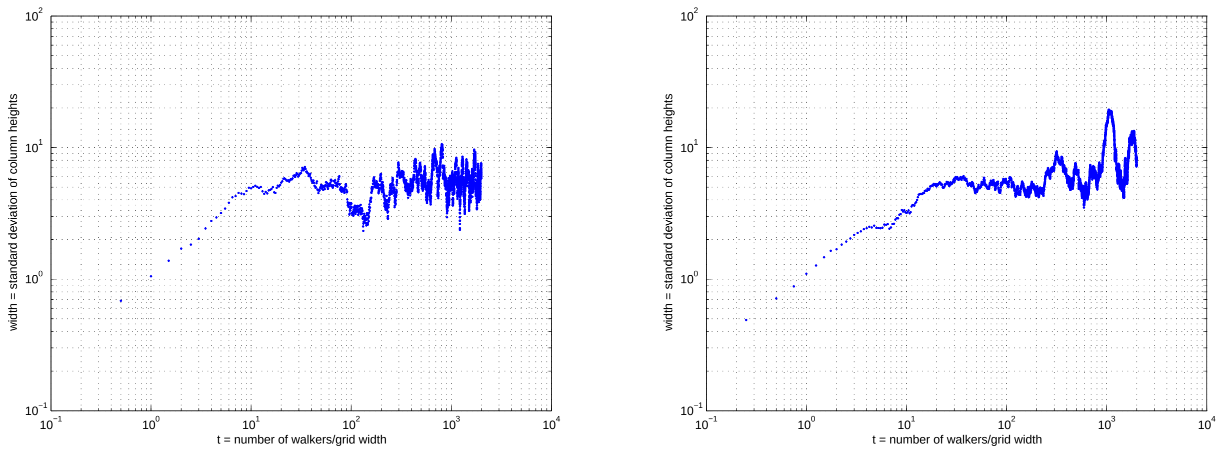
$$\begin{aligned} V(d) &= n^2 - 4n(n-1)p(1-p) - n^2(p-q)^2, \\ &= 4npq. \end{aligned}$$

Recall that the binomial distribution has variance npq. $V(d)$ is greater by a factor of 4. Is this correct? To see that it is, note that corresponding to the values $r = 0, 1 \ldots n$ we have possible values $d = -n \ldots n$. In a sense, the distribution for d is **dilated** by a factor of 2 relative to that for r. Consequently, the variance of the distribution for d is 4 times that for r.

3 Ballistic deposition

3.1 Results of numerical experiments

Figures 2 and 3 show the increase of the interface width as a function of time under ballistic deposition for four different system sizes.



(a) L = 100, n = 200,000 (b) L = 200, n = 400,000

Figure 2: Interface width as a function of time for ballistic deposition

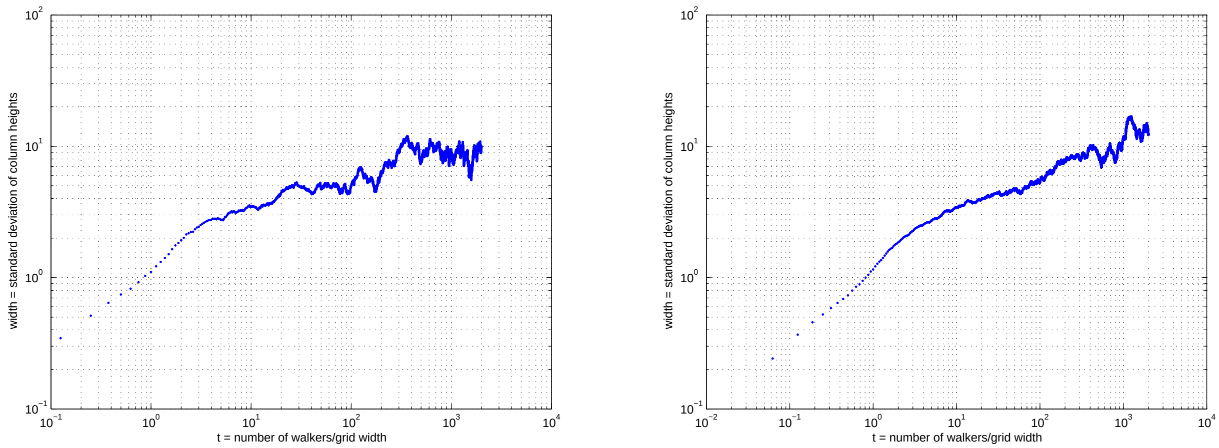
3.2 Parameterization

For small time, the *interface width* appears to grow as some power of the time,

$$w(L,t) \sim t^{\beta}.$$

The interface width stops growing at the *crossover time*, t_x, which depends on the system size,

$$t_x \sim L^z.$$



(a) L = 400, n = 800,000

(b) L = 800, n = 1,600,000

Figure 3: Interface width as a function of time for ballistic deposition

The *saturation width*,

$$w_{\text{sat}}(L) \sim L^{\alpha},$$

also appears to depend on the system size L.

3.3 Comparison with random deposition

For random deposition, note that we have already determined the parameters α, β and z exactly.

- $w(L, n) = \sqrt{npq} \Rightarrow \beta = \frac{1}{2}$.
- $w_{\text{sat}} = \infty \Rightarrow \alpha = \infty$ [The interface width does not saturate]
- $t_x = \infty \Rightarrow z = \infty$ [The interface width does not saturate]

3.4 Scaling

The graphs of interface width vs. time can be made to superimpose by scaling the graphs.

- Rescale the interface width
$$w(L,t) \to \frac{w(L,t)}{w_{\text{sat}}(L)}$$
so that every curve saturates at the same interface width.
- Rescale the time
$$t \to \frac{t}{t_x}$$
so that the interface widths saturate at the same time.

The ability to superimpose the results of the numerical simulations in this way suggests that $w(L,t)/w_{\text{sat}}(L)$ is a function of t/t_x only. That is,

$$\frac{w(L,t)}{w_{\text{sat}}(L)} \sim f\left(\frac{t}{t_x}\right).$$

Replacing w_{sat} and t_x in the equation above with their scaling forms, find that

$$w(L,t) \sim L^{\alpha} f\left(\frac{t}{L^z}\right).$$

3.5 The scaling parameters α, β and z are not independent

[graph]

Approaching the crossover point $(t_x, w(t_x))$ from the left, we have that

$$\lim_{t\to t_x^-} w(t) \sim t_x^{\beta}.$$

Approaching this same point from the right, we have that

$$\lim_{t\to t_x^+} w(t) \sim L^{\alpha}.$$

Together, these relations imply that

$$t_x^\beta \sim L^\alpha.$$

in the neighborhood of the crossover time.

Substituting the scaling relation for the crossover time, $t_x \sim L^z$, into the equation above find that

$$L^{z\beta} \sim L^\alpha$$

or

$$z = \frac{\alpha}{\beta}.$$

3.6 Why does the interface width saturate under ballistic deposition?

Under ballistic deposition, the evolution of the column heights is correlated; the random growth of a locally 'tall' column affects the aggregation of adjacent columns. The influence of tall columns spreads as more particles are added to the interface. Eventually, the influence of locally tall columns spreads across the width of the system at which point it appears that the interface width becomes saturated. *Why* it should become saturated is not clear.

4 Continuum growth equations

For random deposition, we have an exact solution of the microscope (or particle) model. For ballistic deposition, an exact solution is not known.

When an exact solution cannot be found for a microscopic model, it is sometimes helpful to look for a *continuous model* that approximates the particle description. The hope is that the continuum description will be equivalent to the microscopic description in predicting *quantities of interest*, e.g., the interface width as a function of time. We cannot hope that the continuum model will reproduce all aspects of the discrete/particle model.

4.1 Stochastic growth equation

To test this approach, we first attempt to describe the time-evolution of the interface height under *random deposition* using a **stochastic growth equation**.

A general equation describing the accumulation of particles on a surface under random deposition is

$$\frac{\partial h(x,t)}{\partial t} = \Phi(x,t)$$

where $h(x,t)$ is the surface height.

It is helpful to break up the function $\Phi(x,t)$ into two terms,

$$\Phi(x,t) = F(x) + \eta(x,t)$$

where $F(x)$ is the *average number of particles per time* arriving at x and $\eta(x,t)$ represents the *random fluctuations* (or *noise*) in the number of particles arriving per time at x at time t.

The evolution equation for the surface height becomes

$$\frac{\partial h(x,t)}{\partial t} = F(x) + \eta(x,t).$$

4.2 The noise term

$\eta(x,t)$ is a *stochastic process*, a parameterized family of *random variables*. The parameter is usually interpreted, as here, as the time t.

We assume that the *ensemble average* (the expected value) of this stochastic process is zero,

$$\langle \eta(x,t) \rangle = 0,$$

i.e., we assume that the average value of $\eta(x,t)$, taken over many *realizations* of the stochastic process (in the limit as this number $\to \infty$), is zero.

We also assume that the *second moment* (the variance or covariance) of $\eta(x,t)$ is given by

$$\langle \eta(x,t)\eta(x',t') \rangle = D\delta(x-x')\delta(t-t'),$$

where D is a constant equal to the second moment of η when $x = x'$ and $t = t'$. This relation asserts that the noise term has no correlations in space or time except when $x = x'$ and $t = t'$ (processes are always self-correlated).

4.3 Integration

Integrating the stochastic differential equation, find that

$$h(x,t) = F(x)t + \int_0^t d\xi\, \eta(x,\xi).$$

Because $\eta(x,t)$ is a stochastic process, the integration we've just performed is not meaningful if we interpret the integral in the Riemann sense. We will overlook this relatively difficult piece of mathematics and proceed.

Note that because $\eta(x,t)$ is a stochastic process, so too is its integral, whereas $F(x)$ and t are deterministic.

To find the average height of the surface as predicted by the model, we take the ensemble average of both sides of the equation above.

$$\begin{aligned}\langle h(x,t)\rangle &= \langle F(x)\,t\rangle + \langle \int_0^t d\xi\, \eta(x,\xi)\rangle,\\ &= F(x)\,t + \int_0^t d\xi\, \langle \eta(x,\xi)\rangle,\\ &= F(x)\,t.\end{aligned}$$

To find the interface width predicted by the model, recall that

$$\begin{aligned}w^2(t) &= V(h),\\ &= E[(h - E(h))^2],\\ &= E[h^2] - E[h]^2.\end{aligned}$$

To find $E[h^2]$, square the solution to the differential equation,

$$h(x,t)h(x,t) = F^2t^2 + 2Ft\int_0^t d\xi\, \eta(x,\xi) + \int_0^t d\xi\, \eta(x,\xi)\int_0^t d\xi'\, \eta(x,\xi'),$$

and take the ensemble average of both sides,

$$\begin{aligned}
\langle h(x,t)^2\rangle &= \langle F^2t^2\rangle + \langle 2Ft\int_0^t d\xi\,\eta(x,\xi)\rangle + \langle \int_0^t d\xi\,\eta(x,\xi)\int_0^t d\xi'\,\eta(x,\xi')\rangle,\\
&= F^2t^2 + 2Ft\int_0^t d\xi\,\langle\eta(x,\xi)\rangle + \int_0^t\int_0^t d\xi\,d\xi'\,\langle\eta(x,\xi)\eta(x,\xi')\rangle,\\
&= F^2t^2 + D\int_0^t\int_0^t d\xi\,d\xi'\,\delta(\xi-\xi'),\\
&= F^2t^2 + D\int_0^t d\xi,\\
&= F^2t^2 + Dt.
\end{aligned}$$

It follows that

$$w^2(t) = F^2t^2 + Dt - F^2t^2 = Dt.$$

Hence, $w(t) \sim \sqrt{t}$ and it follows that for random deposition the scaling parameter $\beta = 1/2$.

5 Review

- From numerical experiments, we know that under ballistic deposition a graph of the log of the interface width against the log of time shows the interface width initially grows linearly, then saturates at the crossover time, t_x. .
- Comparing the growth of the interface width for different system sizes, L, suggests that ballistic deposition may be characterized by a three *scaling parameters*, α, β and z such that

$$\begin{aligned} w(L,t) &\sim t^{\beta}, \\ t_x &\sim L^{z}, \\ w_{\text{sat}}(L) &\sim L^{\alpha}. \end{aligned}$$

- Under *random deposition*, these parameters are $\beta = 1/2$, $\alpha = \infty$ and $z = \infty$.
- Rather than study the microscopic or particle model for random deposition directly, we can replace it by a *continuum model*. The hope is that the continuum model is more tractable analytically and that it models the quantities of interest (here, the time evolution of the interface width). A stochastic differential equation modelling random deposition is

$$\frac{\partial h}{\partial t} = F(x) + \eta(x,t),$$

  where $\eta(x,t)$ is a stochastic process.
- We solved this stochastic differential equation assuming that the stochastic process, $\eta(x,t)$, satisfies

$$\begin{aligned} \langle \eta(x,t) \rangle &= 0, \\ \langle \eta(x,t)\eta(x',t') \rangle &= D\delta(x-x')\delta(t-t'). \end{aligned}$$

- We found that the solution to the stochastic differential equation predicts that the interface width increases as the square root of the time, in agreement with the solution to the (exact) particle model,

$$w(t) \sim t^{1/2}.$$

6 The Kardar-Parisi-Zhang (KPZ) equation

Numerical experiments with the ballistic deposition model suggest that the scaling parameters for ballistic deposition are

$$\alpha = 0.47 \pm 0.02,$$
$$\beta = 0.33 \pm 0.006.$$

The relation between scaling parameters we deduced last time,

$$z = \frac{\alpha}{\beta},$$

then predicts that $z = 1.42$ for ballistic deposition.

A continuum model for ballistic deposition is given by the *KPZ equation.* In one spatial dimension the KPZ equation is given by

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2} \left(\frac{\partial h}{\partial x} \right)^2 + \eta(x, t).$$

The KPZ model predicts that the scaling parameters are

$$\alpha = 1/2,$$
$$\beta = 1/3,$$
$$z = 3/2,$$

in one spatial dimension.

Because the ballistic deposition and KPZ models appear to have the same scaling behavior, they are said to belong to the same **universality class**.

6.1 Symmetry constraints

To 'discover' the KPZ equation, we begin as we did for random deposition by writing down a very general differential equation describing the growth of an interface,

$$\frac{\partial h}{\partial t} = G + \eta(x, t),$$

where G is, in general, some function of h, x, t, $\frac{\partial h}{\partial x}$, $\frac{\partial^2 h}{\partial x^2}$, etc.

Symmetry constrains the form of the continuum equation modeling ballistic deposition.

- **Invariance under translations in time.** The form of the equation governing the growth of the interface should not depend on where we set the origin of time. Consequently, the evolution equation must be invariant under the transformation $t \to t + \delta_t$.

  This symmetry implies that the evolution equation *cannot include any explicit time dependence in the function G.* Consider, for example, $G(k, x, t) = t$ and $\eta(x, t) = 0$. In this case, we have that
  $$\frac{\partial h}{\partial t} = t.$$
  Let $\bar{t} = t + \delta_t$. Then
  $$\frac{\partial}{\partial t} = \frac{\partial}{\partial \bar{t}} \frac{\partial \bar{t}}{\partial t}$$
  and the evolution equation for the surface height becomes
  $$\frac{\partial h}{\partial \bar{t}} = \bar{t} - \delta_t$$
  which is not of the same form as the original equation.

  Note that unlike the function t, the function $\partial h / \partial t$ is invariant under time translation.

- **Invariance under translation in the direction perpendicular to the growth direction.** The evolution equation should not depend on the coordinate x. That is, the equation should be invariant under the translation $x \to x + \delta_x$.

  This symmetry *excludes explicit dependence of the function G on x.* All partial derivatives with respect to x, however, are invariant under this translation.

- **Invariance under translation along the growth direction.** The evolution equation should not depend on where $h = 0$ is defined. Consequently, the evolution equation should be invariant under the translation $h \to h + \delta_h$.

  This symmetry *excludes explicit dependence of the function G on h.* It does permit all derivatives of h with respect to x to appear in the continuum equation including $\partial h / \partial x$, $\partial^2 h / \partial x^2$, etc.

- **Invariance under inversion about the growth direction.** The evolution equation should be invariant under the transformation $x \to -x$. This requirement *precludes the presence of terms such as $\partial h / \partial x$ which change sign under this transformation.*

6.2 Symmetry considerations applied to the continuum model for random deposition

Under *random deposition*, an additional symmetry is present that is *not* present for the ballistic deposition model. For random deposition, the fluctuations in the interface height are symmetric *about the mean interface height.* That is, from the point of view of an observer moving with the mean surface height, the evolution equation should be invariant under the transformation $h \to -h$.

Because of the presence of the $\partial h/\partial t$ term in the evolution equation, this requirement rules out terms that include *even powers* of h such as $(\partial h/\partial x)^2$. This last term, however, does appear in the KPZ equation modelling ballistic deposition.

Altogether, the symmetry constraints imply that the function $G(h, x, t)$ in the continuum model for random deposition should not depend to lowest order on x, t, h, $\partial h/\partial x$ and $(\partial h/\partial x)^2$. Hence, for random deposition, we have that

$$\frac{\partial h}{\partial t} = G + \eta(x, t).$$

Note that $\partial^2 h/\partial x^2 \to -\partial^2 h/\partial x^2$ under $h \to -h$ and hence is excluded.

6.3 Lateral growth of the interface under BD

[Figure]

Under random deposition, the interface grows in only one direction. By contrast, under ballistic deposition, the interface grows in the direction of its *local normal.* If the interface grows at a rate v, the change in height of the interface, δh, can be found using the Pythagorean theorem.

$$\begin{aligned}\delta h &= \left[(v\delta t)^2 + \left(v\delta t\frac{\partial h}{\partial x}\right)^2\right]^{1/2}, \\ &= v\delta t\left[1 + \left(\frac{\partial h}{\partial x}\right)^2\right]^{1/2}.\end{aligned}$$

In the case that $\frac{\partial h}{\partial x} \ll 1$, we can expand the equation above to obtain

$$\frac{\partial h(x,t)}{\partial t} = v + \frac{v}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \ldots$$

This suggests that the term $\left(\frac{\partial h}{\partial x}\right)^2$ must be included in the continuum growth equation for ballistic deposition. Recall that this term must be *excluded* for random deposition.

6.4 Review

- The KPZ equation is a continuum model for ballistic deposition,

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \eta(x,t).$$

- The KPZ equation can be 'discovered' by examining the symmetries that must be obeyed by any model of ballistic deposition. For example, the KPZ equation should be invariant under time translation, $t \to t + \delta_t$, spatial inversion, $x \to -x$, etc.

- The nonlinear term, $\left(\frac{\partial h}{\partial x}\right)^2$, that appears in the KPZ equation is required for *any model* of interface growth in which the interface growth occurs *normal* to the interface.

- The scaling parameters implied by the KPZ equation can be shown to equal $\alpha = 1/2$, $\beta = 1/3$ and $z = 3/2$. These values closely match those found numerically for ballistic deposition. Consequently, the KPZ and ballistic deposition models are said to belong to the same *universality class.*

6.5 Higher order terms are negligible in the hydrodynamic limit

We would like to show that in the *hydrodynamic limit* higher order terms allowed by symmetry but not included in the KPZ equation are small relative to those retained. As an example, we show that in this limit the term $\partial^4 h/\partial x^4$ would make a negligible contribution relative to that made by $\partial^2 h/\partial x^2$.

Earlier we deduced, on the basis of numerical experiments, that graphs of the interface height, $h(x,t)$, as a function of time for different system sizes can be mapped one to another by an *anisotropic transformation* (*not* isotropic, i.e., differing in the different spatial dimensions). That is, we deduced that the interface height is *invariant* under this transformation.

This invariance implies the following. If we rescale the interface horizontally by $x \to bx$, we must have that

$$L \to bL.$$

Because we assumed that $w_{\text{sat}} \sim L^\alpha$, under the rescaling in x, we must have that

$$w_{\text{sat}} \to b^\alpha L^\alpha.$$

This is to say that

$$w_{\text{sat}} \to b^{\alpha} w_{\text{sat}}.$$

In order that the transformation be invariant, we therefore must have that

$$h \to b^{\alpha} h.$$

In short, a rescaling in x, $x \to bx$ must be accompanied by a rescaling in h, $h \to b^{\alpha}h$ for there to be scale invariance.

The continuum growth equation modeling ballistic deposition must be invariant under these transformations as well. Specifically, if $x' = bx$ and $h' = b^{\alpha}h$, it follows that

$$\begin{aligned} \frac{\partial h'}{\partial x'} &= \frac{\partial h'}{\partial h}\frac{\partial h}{\partial x}\frac{\partial x}{\partial x'}, \\ &= b^{\alpha-1}\frac{\partial h}{\partial x}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 h'}{\partial x'^2} &= \frac{\partial}{\partial x}\left(b^{\alpha-1}\frac{\partial h}{\partial x}\right)\frac{\partial x}{\partial x'}, \\ &= b^{\alpha-2}\frac{\partial^2 h}{\partial x^2}. \end{aligned}$$

Therefore,

$$\frac{\partial^2 h}{\partial x^2} \to b^{\alpha-2}\frac{\partial^2 h}{\partial x^2}.$$

Similarly,

$$\frac{\partial^4 h}{\partial x^4} \to b^{\alpha-4}\frac{\partial^4 h}{\partial x^4}.$$

We are interested in the properties of the growing interface in the *hydrodynamic limit*. This limit measures the behavior of the interface height function as $t \to \infty$ (long-time) and $x \to \infty$ (long-distance) limits. The hydrodynamic limit reveals properties of the interface when that interface is already well established ($t \to \infty$) and at a length scale large relative to the particle (column) size ($x \to \infty$).

In the hydrodynamic limit, $b \to \infty$, $b^{\alpha-4}$ goes to zero much more quickly than does $b^{\alpha-2}$. Consequently, in this limit $\frac{\partial^4 h}{\partial x^4}$ is negligible relative to $\frac{\partial^2 h}{\partial x^2}$. (Recall that $\alpha = 0.47 \pm 0.02$ from numerical experiments).