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Math 310
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1. One can figure this out easily using the addition principle; more specifically, from Proposition 10.3.2, which states

$$|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cup Y \cup Z| \qquad (1)$$

   We are given all of the quantities on the right hand side of this equation in the problem's setup:

| | | | | |
|---|---|---|---|---|
| Reasoning | = | $\lvert R\rvert$ | = | 129 |
| Algebra | = | $\lvert A\rvert$ | = | 129 |
| Calculus | = | $\lvert C\rvert$ | = | 129 |
| Reasoning and Algebra | = | $\lvert R \cap A\rvert$ | = | 85 |
| Reasoning and Calculus | = | $\lvert R \cap C\rvert$ | = | 89 |
| Algebra and Calculus | = | $\lvert A \cap C\rvert$ | = | 86 |
| Reasoning, Algebra and Calculus | = | $\lvert R \cap A \cap C\rvert$ | = | 54 |

   Replacing with our known values, we find that the number of students in R, A, or C (those that found one of these three courses acceptable) is 181. Since we are given there are 182 students in the department, that means there is one sad and bitter individual that enjoys none of these courses. Schade.

2. We will start, as suggested, by assuming that $X$ is a finite set. Being a finite set, it must have a cardinality, which we will simply label $n$. Given that $f$ restricts to give an injection $\mathbf{N}_{\mathrm{n+1}} \to \mathrm{X}$, we see that the cardinality of $\mathbf{N}_{\mathrm{n+1}}$ is greater than $n$.

3. The divisors of 126 are {1,2,3,6,7,9,14,18,21,42,63}, and the divisors of 160 are {1,2,3,4,5,6,9,10,12,15,18, 20,30,36,45,60,90}. From a comparison of these sets, we see that the greatest common divisor of 128 and 160 is 18. Joy.

4. This problem will be solved in two discrete steps. The first part involves us proving that there necessarily exist two elements of $A$, $a$ and $b$ which share a greatest odd divisor. To establish this, we create a function $f(x)$ which gives the greatest odd divisor of $x$. As the set $A$ is constructed from $\mathbf{N}_{\mathrm{2n}}$ and has cardinality $n+1$, we know that $f$ must map to a codomain smaller than $A$, as it returns only positive odd numbers necessarily smaller than $2n$, of which there are at most $n$. Now we have a function $f : A \to X$, where $X = \{a \in \mathbf{N}_{\mathrm{2n}} | \exists \mathrm{q}(\mathrm{a} = 2\mathrm{q} + 1)\}$, and, as we know, the cardinality of $X$ is $n$. This means we can pply the pigeonhole principle and establish that $f$ is not an injection and that for some $a$ and $b$, $f(a) = f(b)$, meaning that some $a$ and $b$ share a common greatest odd factor. Sadly, I am not clever enough to complete this proof- I cannot determine the connection between sharing a greatest odd factor and being divisible by each other.

5. We will begin by defining this set of 10 integers less than 107 as the set $A = \{a_1, a_2, a_3, ...a_10\}$, where $a_n < 107$ for all $n$. We wish to prove that for any such set, there will be two distinct subsets of $A$ where the sum of the elements is identical. The set of possible combinations of A is the power set of $A$, $\wp(A)$. From here, we can define a function $f(P) = \sum_{i \in P} i$ which maps $\wp(A) \to X$, where X is the set of possible sums. As there are 10 elements in $A$, the power set has $2^10$ elements, meaning there are 1024 unique combinations of the elements of A, including the empty set. We now examine the bounds on the sums we can create from these elements. Given that A is to be constructed of ten unique positive integers less than to 107, the largest sum we could possibly construct is $\sum_{i=97}^{106} = 1015$. Given this, we can say that the cardinality of X is at most 1016, including the empty set. Even with this, the cardinality of $\wp(A)$ is greater, and thus, by the pigeonhole principle, $f$ is not an injection, and hence, necessarily, there are two subsets of $A$, $a$ and $b$ that sum to the same value. **Q.E.D.**

6. (a) Given that there are no limitations on ordering here, there are 9! ways to order these books.

(b) This is essentially a question of separate ordering. There are 4! ways to order the English books and 5! ways to order the Math books, so there are $5! \cdot 4!$ ways to order the books.

(c) To think about this one, we can treat the collection of Math books as a single English book in our initial analysis- we have 5 objects, so there are 5! ways to order this collection. However, there are 5! ways to order this "special" object, the Math books, leading to a cardinality of orderings $5! \cdot 5!$.

(d) As there are 5 math books, there are $\binom{5}{2}$ ways to pick 2. 4 English books, so $\binom{4}{2}$ ways to pick these. Total is $\binom{5}{2} \cdot \binom{4}{2}$.