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Math 310
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1. (a) *Prove by contradiction that there does not exist a smallest positive real number.*

   Let us start by proposing that there is a smallest real number k, that is, $k < n$ for all positive real numbers n. Since the division operation is closed to real numbers as long as the divisor is not zero, we can assert the existence of another positive real number j, where $j = k/2$. Any positive number divided by 2 is necessarily smaller, so from this, we can assert $j < k$, which contradicts our proposition that k is the smallest of the positive reals, so the proposition must be false. Q.E.D.

   (b) *Prove by induction on n that, for all positive integers n, 3 divides $4^n + 5$.*

   The proposition $4^n + 5 = 3i$ where i is an integer holds when $n = 1$ –

   $$4^1 + 5 = 9 = 3 \cdot 3$$

   Suppose now as inductive hypothesis that $4^k + 5 = 3i$ where i is an integer holds for some k. It follows that $4^{k+1} + 5 = 3j$ where j is also an integer, then, must also hold. $4^{k+1} = 4 \cdot 4^k + 5$, therefore, since by the inductive hypothesis,

   $$4^k + 5 = 3i \Rightarrow 4^k = 3i - 5,\ 4 \cdot 4^k + 5 = 4(3i - 5) + 5 = 12i - 15 = 3j \Rightarrow 4i + 5 = j$$

   . Thus, j must be an integer, as the operations of multiplication and addition are closed to integers, and all the terms on the left-hand side of the previous equality are integers. This proves the inductive step. Hence, by induction, 3 divides $4^n + 5$ for all $n \geq 1$. Q.E.D.

   (c) *Prove by induction on n that $n! > 2^n$ for all integers n such that $n \geq 4$.*

   The proposition $n! > 2^n$ holds for $n = 4$ –

   $$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24 > 16 = 2^4$$

   Suppose as an inductive hypothesis that $k! > 2^k$. It follows that $(k+1)! > 2^{k+1}$, as $(k+1)! = (k+1) \cdot k!$, and $2^{k+1} = 2 \cdot 2^k$. As we are given by the induction hypothesis that $k! > 2^k$, and $k + 1$ is larger than 2 given that we are bound to $n \geq 4$, we can assert that this inequality also holds for $k + 1$. Hence, by induction, $n! > 2^n$ for all $n \geq 4$. Q.E.D.

   (d) *Prove by induction on n that $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all positive integers n.*

   We can show that the base case of $n = 1$ holds:

   $$\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}$$

   Suppose as an inductive hypothesis that $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$. From this statement, it follows that $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{k+2}$.

   $$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{1}{(k+1)(k+2} + \sum_{i=1}^{k} \frac{1}{i(i+1)}$$

   and

   $$\frac{k+1}{k+2} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{k^2+2k}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

   From these two lines of algebraic manipulation, we obtain the equality $\frac{1}{(k+1)(k+2} + \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$. By the induction hypothesis, we have $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$, so this statement must be true. Hence, by induction, $\sum_{i=1} n \frac{1}{i(i+1)} = \frac{n}{n+1}$ for all $n > 0$. Q.E.D.

2. This argument is begging the question- it assumes that n is in fact the largest integer before proving it when it asserts that $n^2 < n$- note that this is not part of any induction hypothesis!

   This argument is very close to a *reductio ad absurdum* argument proving that there can be no largest integer, as for any integer n, there must exist n^2 which is necessarily larger than n.

3. This is an example of incorrectly applying the induction hypothesis and again begging the question. The error is precisely in the following line:

   > "...On the other hand, all the cows in the subset $\{C_2, \ldots, C_{n+1}\}$ must also have the same color..."

   This statement simply serves to assert *without proof* that C_{n+1} is also a cow of the same color as the rest. The induction hypothesis says nothing at all about specific sets such as the one above- it only applies to the general set $\{C_1, \ldots, C_k\}$, where k is some integer. Given this, we can make no statements about a set missing the first element, and although this set is of the same size as the one proposed int he induction hypothesis, *it is not the same set.*

4. > *Let m and n be any integers. Show that the product $m \cdot n$ is odd if and only if both m and n are odd.*

   For this proof, I will make use of the following definitions of 'even' and 'odd':

   > A number n is even if and only if $n = 2k$ is true for some integer k.
   >
   > A number n is odd if and only if $n = 2k + 1$ is true for some integer k.

   Suppose it is not the case that if m and n are odd, then $m \cdot n$ is odd, or, in other words, $m \cdot n$ is even, or $m \cdot n = 2k$ for some integer k. Since m and n are both odd, they can each be expressed in the form $2k + 1$ for some k – to avoid confusion, we will use the variables i and j, both integers, like so:

   $$m \cdot n \Rightarrow (2i + 1)(2j + 1) = 4ij + 2i + 2j + 1 = 2(2ij + i + j) + 1 = 2k$$

   As multiplication and addition are closed to integers, it is clear that $2ij + i + j$ must also be an integer (we will call it p), so we can write $2p + 1 = 2k$. This statement is clearly false, as k and p are both integers (Intuitively, this statement is attempting to say that an even number is also an odd number, which greatly offends the sensibilities of the rational mind). Hence, a contradiction, therefore the supposition is false, thus, it must be the case that if m and n are both odd, then their product must be odd.
   Let us further suppose that it is not the case that if a product of two integers $m \cdot n$ is odd that the factors m and n are also odd. There are two cases here – in one case, one of the variables is odd and one is even. We can express this as $2i(2j + 1) = 2k + 1$. This can be rearranged into $2(2ij + i) = 2k + 1$. Again, due to the closure of multiplication and addition to integers, we can assert that $2ij + i$ is also an integer p, and have $2p = 2k + 1$, which is a contradiction, as $p = \frac{2k+1}{2}$ cannot be an integer.
   There is one last case where both m and n are even that we must eliminate before we can assert our second proposition as truth. This proceeds much in the same way –

   $$m \cdot n = 2k + 1 = 2i \cdot 2j = 2(ij) = 2p$$

   Again we reach $2p = 2k + 1$, and have the contradiction $p = \frac{2k+1}{2}$. As m and n cannot be both even, nor can one be even and one odd, they must both be odd to have an odd product.
   Finally, as we have proven that if m and n are odd then $m \cdot n$ must be odd and that if $m \cdot n$ is odd then both m and n are odd, we can state that $m \cdot n$ is odd if and only if both m and n are odd. Q.E.D.