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 Amath 352 - B. Bale  
 Homework 7

1. (a) First, we set up the system:

```
>> load income.dat;
>> x = income(: , 1);
>> y = income(: , 2);
>> for j = 1:6
    for k = 1:6
        A(j,k) = x(j)^(6-k);
    end
end
>> A
```

A =

1.0e+016 \*

Columns 1 through 3

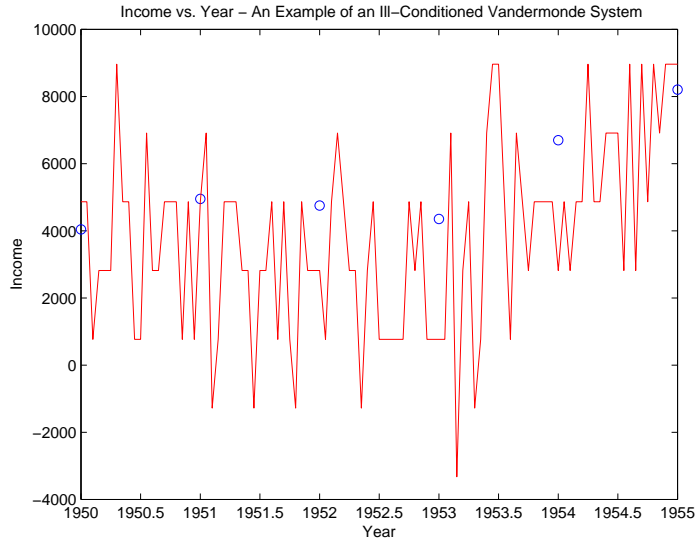
2.81950621875000	0.00144590062500	0.00000074148750
2.82674314055348	0.00144886885728	0.00000074262884
2.83399491493560	0.00145184165724	0.00000074377134
2.84126156474650	0.00145481902957	0.00000074491502
2.84854311285970	0.00145780097895	0.00000074605987
2.85583958217219	0.00146078751006	0.00000074720589

Columns 4 through 6

0.00000000038025	0.00000000000020	0.00000000000000
0.00000000038064	0.00000000000020	0.00000000000000
0.00000000038103	0.00000000000020	0.00000000000000
0.00000000038142	0.00000000000020	0.00000000000000
0.00000000038181	0.00000000000020	0.00000000000000
0.00000000038220	0.00000000000020	0.00000000000000

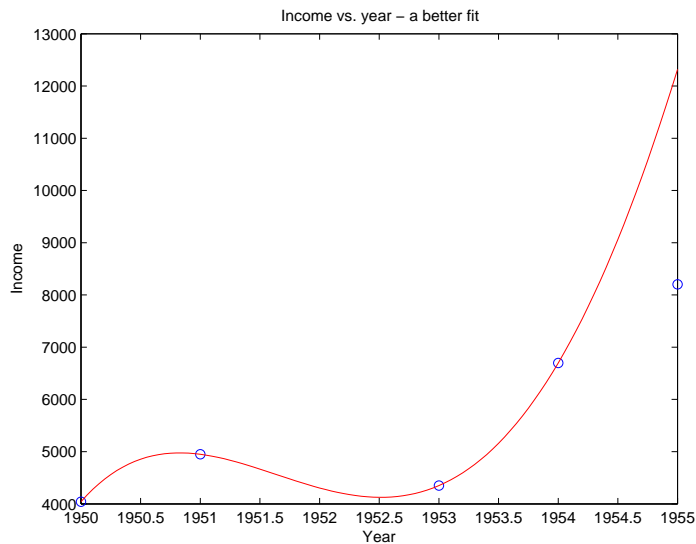
We can see right here that this is not going to end well. This matrix is ill-conditioned, and we have two very close data points in  $y$ .

```
(b) >> plot(x,y,'o')
>> c = A\y;
Warning: Matrix is close to singular or badly scaled.
Results may be inaccurate. RCOND = 2.705366e-33.
>> fx = linspace(1950, 1955,1001);
> fy = c(1)*fx.^5+c(2)*fx.^4+c(3)*fx.^3+c(4)*fx.^2+c(5)*fx+c(6);
>> hold on; plot(fx,fy,'r')
>> hold off; plot(x,y,'o');
>> fx = linspace(1950, 1955,101);
>> fy = c(1)*fx.^5+c(2)*fx.^4+c(3)*fx.^3+c(4)*fx.^2+c(5)*fx+c(6);
>> hold on; plot(fx,fy,'r')
>> xlabel('Year');ylabel('Income');
>> title('Income vs. Year - An Example of an Ill-Conditioned Vandermonde System');
```



As predicted, this is very ugly.

```
(c) >> x = [1950 1951 1953 1954 1965];
>> y = [y(1) y(2) y(4) y(5) y(6)]';
>> A = zeros(5,5);
>> for j=1:5
    for k=1:5
        A(j,k) = x(j)^(5-k);
    end
end
>> c = A\y;
>> fx = linspace(1950, 1955,101);
>> fy = c(1)*fx.^4+c(2)*fx.^3+c(3)*fx.^2+c(4)*fx+c(5);
>> hold off; plot(x,y,'o');hold on;plot(fx,fy,'r');
>> xlabel('Year');ylabel('Income');title('Income vs. year - a better fit');
```



This is a much better fit.

```
(d)
2. >> A
```

A =

```
    1    1    1
    4    2    1
   16    4    1
```

>> c=A\y

c =

```
-0.6667
 3.0000
-0.3333
```

So the interpolating polynomial is:

$$P_2(x) = -\frac{2}{3}x^2 + 3x - \frac{1}{3}$$

The general form of a second order Lagrange polynomial is as follows:

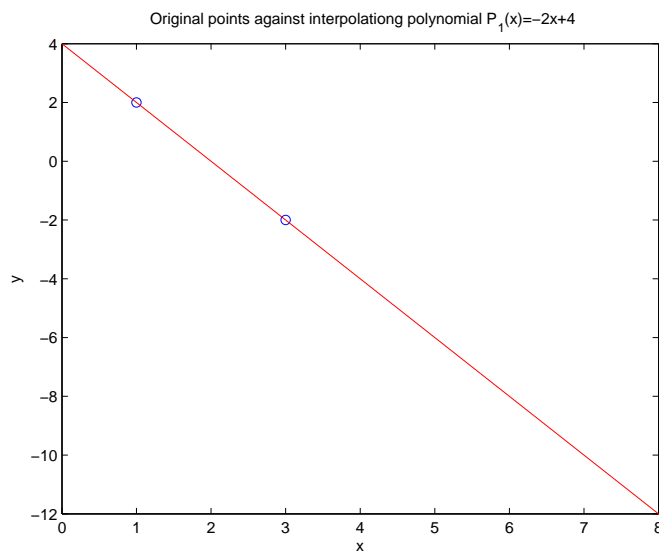
$$\begin{aligned} P_2(x) &= f(x_1) \frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_3-x_2)} + f(x_2) \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} + f(x_3) \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} \\ &= 2 \frac{(x-2)(x-4)}{(1-2)(4-2)} + 3 \frac{(x-1)(x-4)}{(2-1)(2-4)} + \frac{(x-1)(x-2)}{(4-1)(4-2)} \\ &= -\frac{2}{3}x^2 + 3x - \frac{1}{3} \end{aligned}$$

(b) For the newton form, it goes like this:

$$\begin{aligned} P_2(x) &= f(x_1) + F[x_1, x_2](x-x_1) + F[x_1, x_2, x_3](x-x_1)(x-x_2) \\ &= 2 + \frac{3-2}{2-1}(x-1) + \frac{\frac{1-3}{4-2} - \frac{3-2}{2-1}}{4-1}(x-1)(x-2) \\ &= -\frac{2}{3}x^2 + 3x - \frac{1}{3} \end{aligned}$$

So, they are all the same polynomial.

3. (a) Our interpolating polynomial is  $P_1(x) = -2x + 4$ .



(b)

$$P_2(x) = 2 - \frac{-2-2}{3-1}(x-1)$$

4. (a) The general form of a Newton basis first order polynomial is as follows-

$$P_1(x) = c_1 + c_2(x - x_1)$$

Using divided differences, we can solve for the specific form-

$$c_1 = F[x_1] = f(x_1) - h \quad c_2 = F[x_1, x_2] = \frac{f(x_2) - f(x_2 - h)}{x_2 - x_1}$$

$$P_1(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1)$$
$$P_1'(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_2) - f(x_2 - h)}{h}$$

(b) Second verse the same as the first.

$$P_2(x) = c_1 + c_2(x - x_1) + c_3(x - x_1)(x - x_2)$$

$$c_3 = F[x_1, x_2, x_3] = \frac{F[x_2, x_3] - F[x_1, x_2]}{2h} = h(f(x_1) - 2f(x_2) + f(x_3))$$

And, because  $c_1$  and  $c_2$  are the same as in the linear case...

$$P_2(x) = f(x_1) + \frac{f(x_2) - f(x_1)}{h}(x - x_1) + h(f(x_1) - 2f(x_2) + f(x_3))(x - x_1)(x - x_2)$$

Now we find the derivative..

$$P_2'(x) = \frac{f(x_2) - f(x_1)}{h} + h(f(x_1) - 2f(x_2) + f(x_3))((x - x_1) + (x - x_2))$$

$$P_2'(x_2) = \frac{f(x_2) - f(x_1)}{h} + h^2(f(x_1) - 2f(x_2) + f(x_3))$$

(c) The form of a second order Lagrange basis interpolating polynomial is as follows:

$$P_2^{(L)}(x) = f(x_1)L_1 + f(x_2)L_2 + f(x_3)L_3$$

where

$$L_j(x) = \prod_{k=1, k \neq j}^n \frac{x - x_k}{x_j - x_k}$$

$$P_2(x) = f(x_1) \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_3 - x_2)} + f(x_2) \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} + f(x_3) \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

$$P_2'(x) = \frac{f(x_1)}{(x_1 - x_2)(x_1 - x_3)}((x - x_2) + (x - x_3)) + \frac{f(x_2)}{(x_2 - x_1)(x_2 - x_3)}((x - x_1) + (x - x_3)) + \frac{f(x_3)}{(x_3 - x_1)(x_3 - x_2)}((x - x_1) + (x - x_2))$$

$$P_2^{(L)'}(x_2) = -\frac{f(x_1)}{2h} + \frac{f(x_3)}{2h}$$

(d)

$$\begin{array}{lll} f(x_1) = -3.3807 & f(x_2) = -8.3585 & f(x_3) = -15.9558 \\ f'(x_1) = -39.1233 & f'(x_2) = -61.6031 & f'(x_3) = -91.7084 \\ P_1'(x_1) = -49.7783 & P_1'(x_2) = -49.7783 & P_1'(x_3) = -49.7783 \\ P_2'(x_1) = -49.7521 & P_2'(x_2) = -49.8045 & P_2'(x_3) = -49.8568 \\ P_2^{(L)'}(x_1) = 297.6603 & P_2^{(L)'}(x_2) = 104.2951 & P_2^{(L)'}(x_3) = -89.0702 \end{array}$$

Ack. I have clearly done something wrong. However, that was a fucking evil amount of hand calculation. I am not doing it again.

(e) OH JESUS YOU ARE JOKING.

No, seriously, this is too much brute force nonsense. Please don't inflict this assignment on your future students...

$$\begin{array}{lll} f(x_1) = -8.2971 & f(x_2) = -8.3585 & f(x_3) = -8.4203 \\ f'(x_1) = -61.3425 & f'(x_2) = -61.6031 & f'(x_3) = -61.8644 \\ P_1'(x_1) = & P_1'(x_2) = -49.7783 & P_1'(x_3) = -49.7783 \\ P_2'(x_1) = & P_2'(x_2) = & P_2'(x_3) = \\ P_2^{(L)'}(x_1) = & P_2^{(L)'}(x_2) = & P_2^{(L)'}(x_3) = \end{array}$$

(f)