

1. (a)

$$\begin{aligned}
 A &= \begin{bmatrix} 2 & 8 & 6 & 20 \\ 8 & 33 & 29 & 84 \\ 6 & 26 & 30 & 70 \end{bmatrix} & \rightarrow & \begin{bmatrix} 8 & 33 & 29 & 84 \\ 2 & 8 & 6 & 20 \\ 6 & 26 & 30 & 70 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 33/8 & 29/8 & 21/2 \\ 2 & 8 & 6 & 20 \\ 6 & 26 & 30 & 70 \end{bmatrix} \\
 &\rightarrow \begin{bmatrix} 1 & 33/8 & 29/8 & 21/2 \\ 2 & 8 & 6 & 20 \\ 6 & 26 & 30 & 70 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 33/8 & 29/8 & 21/2 \\ 0 & -1/4 & -5/4 & -1 \\ 0 & 5/4 & 33/4 & 7 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 33/8 & 29/8 & 21/2 \\ 0 & 5/4 & 33/4 & 7 \\ 0 & -1/4 & -5/4 & -1 \end{bmatrix} \\
 &\rightarrow \begin{bmatrix} 1 & 33/8 & 29/8 & 21/2 \\ 0 & 1 & 33/5 & 28/5 \\ 0 & -1/4 & -5/4 & -1 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 0 & -118/5 & -63/5 \\ 0 & 1 & 33/5 & 28/5 \\ 0 & 0 & 2/5 & 2/5 \end{bmatrix} & \rightarrow & \begin{bmatrix} 1 & 0 & -118/5 & -63/5 \\ 0 & 1 & 33/5 & 28/5 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
 &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}
 \end{aligned}$$

$$\text{So, } x = \begin{bmatrix} 11 \\ -1 \\ 1 \end{bmatrix}.$$

(b) As $A = CDE$, and C is a lower triangular matrix, D is a permutation matrix, and E is an upper triangular matrix, we have just been given an LU factorization of A . From this point, it is relatively easy to solve this system by solving two triangular matrix equations, $Cx = y$ and $DEy = b$.

2. I'll be using the Doolittle algorithm to perform the LU factorization here. Basically, this entails performing basic row operations on the original matrix in the form of left-multiply transform matrices to reduce A to upper triangular, then multiplying the transform matrices together to come up with a single lower triangular matrix.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{9}{2} & \frac{15}{2} \\ 0 & 0 & 4 \end{bmatrix}$$

3. (a) To solve this, we invert the matrix from how it is given in the book:

$$A = \begin{bmatrix} 80 & 0 & 30 & 10 \\ 0 & 80 & 10 & 10 \\ 16 & 20 & 60 & 72 \\ 4 & 0 & 0 & 8 \end{bmatrix} \qquad b = \begin{bmatrix} 40 \\ 27 \\ 31 \\ 2 \end{bmatrix}$$

and go about solving this system of equations with the matlab backslash command.

```
>> PM = [80 0 30 10; 0 80 10 10; 16 20 60 72; 4 0 0 8];
>> b = [40 27 31 2];
>> PM\b'
```

ans =

```
0.4000
0.3000
0.2500
0.0500
```

Hence, creating a popular paint from these paints of middling quality requires a blend of 40% paint no.1, 30% no.2, 25% no.3, and 5% no.4.

- (b) Well, a direct linear combination is possible, but to create this mix in practice from these paints, we'd need to discover antipaint no.3, a substance which, upon contact with paint no.3, eradicates the paint and itself. As this is not a substance known to science, a large amount of resurces would most likely have to be devoted to its creation. Furthermore, the cancellation of the paint by the antipaint would most likely result in a large amount of energy being produced, the containment of which results in this procedure being neither practical nor marketable.

We get a negative coefficient from the backslash command.

```
>> PM\[ [25 25 25 25] ' )
```

```
ans =
```

```
1.1474
0.3782
-3.0769
2.5513
```

.. and this is a physical impossibility (without the aforementioned antipaint no.3). So, no, this is not something we can make from these paints.

4. (a) From the statements above, we get the following equations-

$$x_1 + x_2 + x_3 + x_4 = 13 \qquad x_1 + x_2 + x_3 + x_4 = 13 \qquad (1)$$

$$x_2 + x_4 - 1 = x_1 + x_3 \qquad x_1 - x_2 + x_3 - x_4 = -1 \qquad (2)$$

$$2x_1 = x_4 \qquad 2x_1 - x_4 = 0 \qquad (3)$$

$$x_3 - 1 = x_2 \qquad x_2 - x_3 = -1 \qquad (4)$$

- (b) Manual row reduction is gross and ugly and for these reasons (not that it's kinda pointless) is it not included on this sheet. yarr!
- Bah, okay, fine, I'll include it.

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 1 & 1 & 1 & 13 \\ 1 & -1 & 1 & -1 & -1 \\ 2 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 13 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 & 0 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 13 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \\
 &\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 & 0 \\ 0 & -1 & 1 & -1/2 & -1 \\ 0 & 1 & 1 & 3/2 & 13 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 1/2 & 1 \\ 0 & 0 & 2 & 1 & 12 \\ 0 & 0 & 0 & -1/2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & -1 & 1/2 & 1 \\ 0 & 0 & 1 & 1/2 & 6 \\ 0 & 0 & 0 & -1/2 & -2 \end{bmatrix} \\
 &\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1/2 & 6 \\ 0 & 0 & 0 & -1/2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1/2 & 0 \\ 0 & 1 & 0 & 1 & 7 \\ 0 & 0 & 1 & 1/2 & 6 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 4 \end{bmatrix}
 \end{aligned}$$

..which means they have 2 hearts, 3 clubs, 4 diamonds and 4 spades. What a card shark we are.

- (c) >> A = [1 1 1 1; 1 -1 1 -1; 2 0 0 -1; 0 1 -1 0];
 >> b = [13 -1 0 -1];
 >> A\b'

```
ans =
```

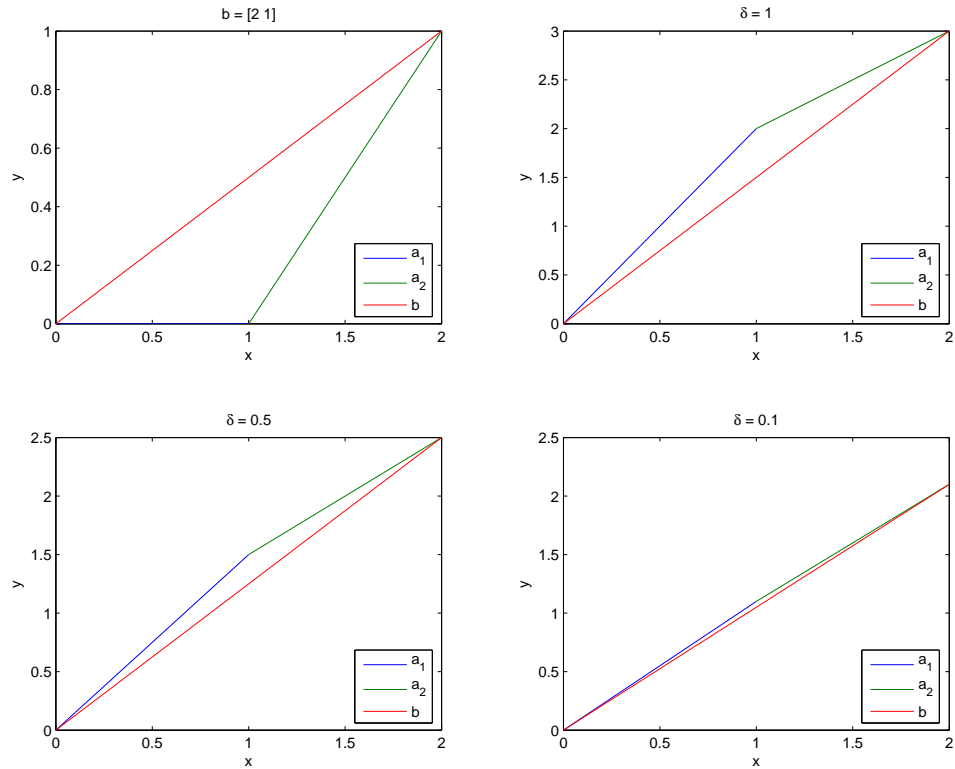
```
2
```

3
4
4

5. (a)

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 - \frac{b_2 - b_1}{\delta} \\ \frac{b_2 - b_1}{\delta} \end{bmatrix}$$

(b) Drawings!



(c) Computing an inverse by hand of a 2x2 matrix is fairly easy.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} 1 & -1 \\ -1 - \delta & 1 \end{bmatrix} = \frac{1}{-\delta} \begin{bmatrix} 1 & -1 \\ -1 - \delta & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{-\delta} & \frac{1}{\delta} \\ 1 + \frac{1}{\delta} & -\frac{1}{\delta} \end{bmatrix}$$

$$\text{cond}(A) = \|A\|_{\infty} \cdot \|A^{-1}\|_{\infty} = \max(2, 2 + \delta) \cdot \left(1 + \frac{1}{\delta}\right)$$

The above equation is simplified if $\delta \geq 0$.

```
>> for dl=[1 .5 .1 1e-10]
cond([1 1; 1+dl 1],inf)
end
```

ans =

9

ans =

12.5000

ans =

44.1000

ans =

4.0000e+10

(d) In the case that $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

With $b = \begin{bmatrix} 1 \\ 1.000001 \end{bmatrix}$, the solution changes. The augmented matrix for the system is $\begin{bmatrix} 1 & 1 & 1 \\ 1 + \delta & 1 & 1.000001 \end{bmatrix}$,

which in RREF form is $\begin{bmatrix} 1 & 0 & 10000 \\ 0 & 1 & -9999 \end{bmatrix}$, meaning $x = \begin{bmatrix} 10000 \\ -9999 \end{bmatrix}$. Matlab's solution is a bit wonky-

```
>> format long e
```

```
>> A\b
```

ans =

9.999999170773692e+03

-9.998999170773692e+03

So, while close, we really get no correct digits for x_1 , and 3 correct digits for x_2 .

I'm having a hard time recognizing the correlation between the condition number of A and the result. However, if we are to take the deviation from the original b as error, we can make the statement that the error seems to be around the order of $\text{cond}(A) \cdot \delta_b$. This matches up about right with the numeric answer and matlab's answer, but is off by a factor of 4. Perhaps this is related to the size of the matrix?