Peter Woodman
0127119
Amath 352 - B. Bale
Homework 2

1. 4. See attached sheet.

   23. For Ax:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix}$$

   The resulting vector has n rows, each requiring n multiplies and $n-1$ adds, so we have $O(n^2)$ operations.

   For $x^T A$:

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} =$$

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & \cdots & a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{bmatrix}$$

   Here we have the exact same scenario, except with columns.

   29.

$$d = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

   One notes that with d, one cannot affect any change in the third element. Thus,to have a basis in $\mathbb{R}^3$, of these vectors, we must have c to affect change in this column.

   39. By hand

   40.

```
>> r = (1/2)*[1 1 1 1]';
>> s = (1/2)*[1 1 -1 -1]';
>> t = (1/2)*[sqrt(2) -sqrt(2) 0 0]';
>> u = (1/2)*[0 0 sqrt(2) -sqrt(2)]';
>> r'*s

ans =

     0

>> [r'*s r'*t s'*t t'*u]

ans =

     0     0     0     0

>> [norm(r) norm(s) norm(t) norm(u)]

ans =

    1.0000    1.0000    1.0000    1.0000
```

   .. so these vectors are orthagonal and normal.

41.
```
>> Q = [r s t u];
>> x1 = [rand rand rand rand];
>> x2 = [rand rand rand rand];
>> x3 = [rand rand rand rand];
>> x4 = [rand rand rand rand];
>>  [norm(x1) == norm(Q*x1') norm(x2) == norm(Q*x2') norm(x3) == norm(Q*x3')
     norm(x4) == norm(Q*x4')]

ans =

     1     1     1     1
```

2. (a) These vectors are linearly independent. There is no way you can scale one to the other. Since there are only two vectors, we don't need to look at combinations.

   (b) These vectors are linearly dependent. One can use Matlab to prove this in two ways- first, we can ask it to find the null space of the matrix $A = [v_1 v_2 v_3]$-

```
>> A = [1 1 2 ; 0 2 4 ; 2 -1 -2]';
>> null(A,'r')

ans =

   -2.0000
    1.5000
    1.0000
```

   As this is not the zero vector, we know that this is not a linearly independent set. One could also ask it to compute the rank of A-

```
>> rank(A)

ans =

     2
```

3. (a) U. As U is a linearly independant 2x2 matrix, U spans $\mathbb{R}^2$.

   V. It will be very hard for me to define the span of V without using a basis, so I will just go ahead and do it. The span of V is a plane in $\mathbb{R}^3$ (as the rank of the matrix V is 2) where any vector inside it is a linear combination of the columns of V.

   (b) U. A basis of the span of U is simply U, as U is linearly independent.

   V. A basis of V would be any two of the columns of V - for example, $[v_1 v_2]$.

   (c) U. As U is linearly independant, the null space of U is $\emptyset$, as finding a vector in the null space is equivalent to finding a non-trivial (i.e. not the zero vector) to the matrix eqation $Ax = 0$, which implies a linear combination.

   V. To find this, we can solve $Ax = 0$. Without any fancy tricks, one can see that

$$2 \cdot \overrightarrow{v}_1 - 1.5 \cdot \overrightarrow{v}_2 - 1 \cdot \overrightarrow{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

```
>> [rank(U) rank(V)]
ans =
     2     2
```

   (d) U. Null space is $\emptyset$, these3 vectors are linearly independent.

```
>> null(U)

ans =

   Empty matrix: 2-by-0
```

V. Null space is the line defined by the vector $x = \begin{bmatrix} -2 \\ \frac{1}{2} \\ 1 \end{bmatrix}$.

```
>> null(V,'r')

ans =

   -2.0000
    1.5000
    1.0000
```

(e) For both, the range is identical to the span, as the span is equal to the column space.

4. We are given the following-

$$
\begin{aligned}
A &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \\
B &= \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} \\
AB &= \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_n \end{bmatrix} = \emptyset
\end{aligned}
$$

This last statement which defines AB states that the result of Ab_x for all x is $\emptyset$. However, since A is nonsingular, the only way this is possible is if all vectors in B are orthogonal to the vectors in A, and this is not possible as they are the same size. Hence, B must be composed entirely of zero vectors, or $B = \emptyset$. QED.

5. The range of A^T is the same thing as the row space of A. In the row method of matrix-vector multiplication, the elements of the product vector are computed by taking the dot-product of the original vector times the corresponding row of the matrix. All vectors x in the null space satisfy $Ax = 0$. Consequentially, to form the zero vector, the results of these dot products must all be zero, meaning that x is orthogonal to the rows in A, or, the columns (range) of A^T.

6. (a)

$$X = \begin{bmatrix} a & 1-a^2 \\ 1 & -a \end{bmatrix} \qquad X^2 = \begin{bmatrix} a & 1-a^2 \\ 1 & -a \end{bmatrix} \begin{bmatrix} a & 1-a^2 \\ 1 & -a \end{bmatrix}$$

$$X^2 = \begin{bmatrix} a^2+1-a^2 & a-a^3-a+a^3 \\ a-a & 1-a^2+a^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, $X^2 = I$ for every a. QED.

(b)

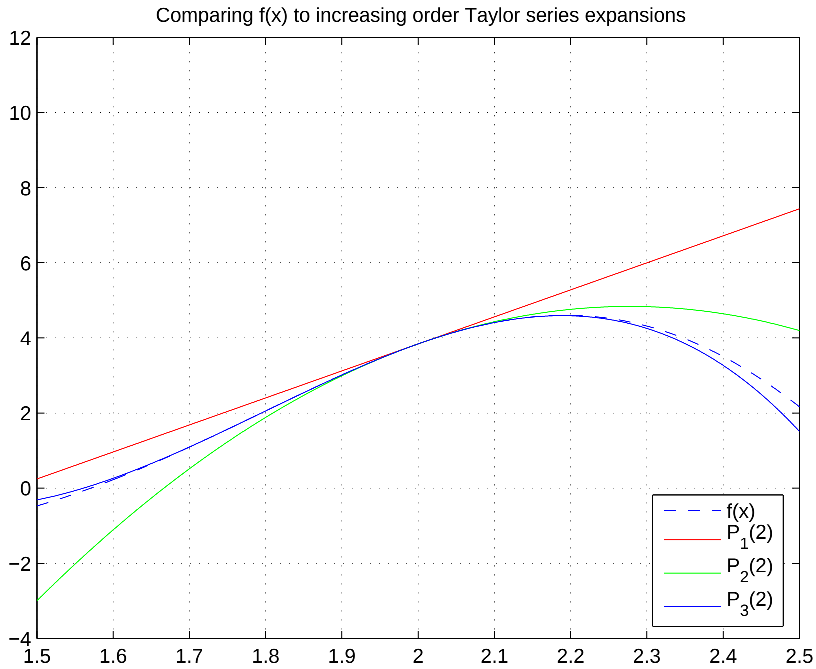
$$X^2 = \begin{bmatrix} b^2 & 0 \\ bc-bc & b^2 \end{bmatrix} = \begin{bmatrix} b^2 & 0 \\ 0 & b^2 \end{bmatrix}$$

So, as long as $b = \pm 1$, $X^2 = I$. QED.

7. (a) Fine.

$$
\begin{aligned}
P_0(x) =& x^2\cos(3x) \\
P_1(x) =& x^2\cos(3x) + 2x\cos(3x) - 3x^2\sin(3x) \\
P_2(x) =& x^2\cos(3x) + 2x\cos(3x) - 3x^2\sin(3x) + 2\cos(3x) - 9x^2\cos(3x) - 12x\sin(3x) \\
P_3(x) =& x^2\cos(3x) + 2x\cos(3x) - 3x^2\sin(3x) + 2\cos(3x) - 9x^2\cos(3x) - 12x\sin(3x) \\
& - 54x\cos(3x) - 18\sin(3x) + 27x^2\sin(3x)
\end{aligned}
$$

(b)



(c) Ok, deal.

(d) It is generally known that the expansion of a Taylor series of a function around a given pivot approaches the value of that function as the number of terms of the Taylor series calculated approaches infinity. Looking at this graph, even with a slightly irregular function such as the one we are working with, we do see a trend where the total difference between the expansion and the actual function is decreasing as we go from P_1 to P_3 quite dramatically- on this particular interval, P_3 is a pretty close approximation for the function itself.